Math 55b: Honors Real and Complex Analysis

Homework Assignment #12 (15 April 2011):
Complex Analysis I

> The shortest path to any truth involving real quantities often passes through the complex plane. —*J. Hadamard* [attributed by several online sources]

You've probably seen some instances of this already (e.g. using complex numbers to describe rotations in the Euclidean plane), and we've done more last term (such as using the Fundamental Theorem of Algebra to describe factorization in $\mathbf{R}[X]$), but complex analysis is the most prominent source of examples. The usual illustration is the to evaluation of real definite integrals via contour integration, and we'll see a lot of that soon, but there will be others including some of the problems below.

First, more about basic properties of analytic functions and contour integrals:

1. (Reflection principles)
   i) Let E be an open set in $\mathbf{C}$, and $E' = \{z \in \mathbf{C} : \bar{z} \in E\}$ (I can't call this "$\overline{E}$" because that looks like topological closure). Prove that $f : E \to \mathbf{C}$ is differentiable if and only if the function $E' \to \mathbf{C}$ defined by $z \mapsto \overline{f(\bar{z})}$ is differentiable. Deduce that if E is an open rectangle or circle symmetric about the real axis and $f(z) \in \mathbf{R}$ for all $z \in E \cap \mathbf{R}$ then $f(\bar{z}) = \overline{f(z)}$ for all $z \in E$.
   ii) Suppose now that $r > 1$ and let E be the annulus $\{z \in \mathbf{C} : 1/r < |z| < r\}$. If $f : E \to \mathbf{C}$ is a differentiable function such that $|f(z)| = 1$ for all z on the unit circle $|z| = 1$, what can you deduce about f?

2. (Green's Theorem for a rectangle; cf. Rudin Thm. 10.45 on pages 282–283) Let γ be the positively oriented boundary of a rectangle $R \subset \mathbf{R}^2$ with sides parallel to the coordinate axes, and let α, β be continuously differentiable functions from an open neighborhood of R to $\mathbf{C}$. Prove that
$$\oint_\gamma (\alpha\, dx + \beta\, dy) = \iint\limits_R \left(\frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y}\right) dx\, dy\,.$$
   Show that (using the standard identification of $\mathbf{R}^2$ with $\mathbf{C}$) this yields the rectangular Cauchy's theorem $\oint_\gamma f(z)\, dz = 0$ if f is *continuously* differentiable as a function of a complex variable on a neighborhood of R. What is $\oint_\gamma \bar{z}\, dz$?

3. Let E be an open set in $\mathbf{C}$, fix some real number B, and let $\mathcal{F}$ be the family of analytic functions $f : E \to \mathbf{C}$ such that $|f(z)| \leq B$ for all $z \in E$. If K is any compact subset of E, prove that $\{f'(z) : f \in \mathcal{F},\, z \in K\}$ is bounded. Deduce that "$\mathcal{F}|_K$ is equicontinuous": for all $\epsilon > 0$ there exists $\delta > 0$ such that $|f(z) - f(z')| < \epsilon$ for all $f \in \mathcal{F}$ and any $z, z' \in K$ such that $|z - z'| < \delta$.

4. For some real numbers r_1, r_2 with $0 < r_1 < r_2$, let A be the annulus $\{z \in \mathbf{C} : r_1 < |z| < r_2\}$, and E an open set containing A. For $r_1 \leq r \leq r_2$ let γ_r be the positively oriented circle $\{re^{i\theta} : 0 \leq \theta \leq 2\pi\}$ in A. Prove that any analytic function $f : E \to \mathbf{C}$ can be written as $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, converging absolutely and uniformly in A, where for each $n \in \mathbf{Z}$ we have
$$a_n = \frac{1}{2\pi i} \oint_{\gamma_r} f(z)\, z^{-n}\, \frac{dz}{z}$$

for any $r \in [r_1, r_2]$.

5. Let $F : \mathbf{R} \to \mathbf{C}$ be a (real-)analytic function such that $F(\theta + 2\pi) = F(\theta)$ for all $\theta \in \mathbf{R}$. Prove that there exist A, E, f as in the previous problem with $r_1 < 1 < r_2$ such that $F(\theta) = f(e^{i\theta})$ for all $\theta \in \mathbf{R}$. Where have you seen the resulting formula for a_n and $F(\theta)$?

6. Prove the following generalization of Liouville's theorem: for any entire function f, there is $r \in \mathbf{R}$ such that $|f(z)| = o(|z|^r)$ as $|z| \to \infty$ if and only if f is a polynomial of degree less than r.

We next introduce functions to and from the Riemann sphere, which can be defined as the "projective line" $\mathbf{P}^1(\mathbf{C}) = (\mathbf{C}^2 - \{(0,0)\})/\mathbf{C}^*$. An analytic map f from some open $E \subset \mathbf{C}$ to $\mathbf{P}^1(\mathbf{C})$ is a pair (f_1, f_2) of analytic functions on E that do not have a common zero in E; another such pair (g_1, g_2) gives the same function if the vectors $(f_1(z), f_2(z))$ and $(g_1(z), g_2(z))$ are proportional for all $z \in E$. You should check that every z has a neighborhood in which f has a representative such that either f_1 or f_2 is the constant function 1. We usually identify f with f_1/f_2, with the understanding that this may take the value ∞ (when $f_2 = 0$). The idea is that $\mathbf{P}^1(\mathbf{C})$ is made up of two copies of $\mathbf{C}$, one represented by vectors $(z_1, 1)$ and the other by $(1, z_2)$, and intersecting on $\mathbf{C}^*$ with $z_2 = z_1^{-1}$. Since the map $z_1 \mapsto z_1^{-1}$ is analytic on $\mathbf{C}^*$ it makes sense to say that an analytic or meromorphic map F *from* $\mathbf{P}^1(\mathbf{C})$ to $\mathbf{C}$ or $\mathbf{P}^1(\mathbf{C})$ is one whose restriction to each of these two copies of $\mathbf{C}$ is analytic or meromorphic respectively.[1]

7. Prove that analytic maps $E \to \mathbf{P}^1(\mathbf{C})$ are precisely meromorphic functions on E together with the constant map ∞. Show that the meromorphic functions on $\mathbf{P}^1(\mathbf{C})$ are precisely the rational functions (note that the rational functions of z_1 are the same as the rational functions of $z_2 = z_1^{-1}$ so this is well-defined). [Use the previous problem, which indeed is the special case of a meromorphic function with no poles except possibly at ∞.]

8. Let $A, B \in \mathbf{C}[z]$ be polynomials such that B has distinct roots $z_1, \ldots, z_n$. Let ω be the differential $(A(z)/B(z))\, dz$ on $\mathbf{C} - \{z_1, \ldots, z_n\}$. Show that the residue of ω at each z_j is $A(z_j)/B'(z_j)$. Conclude that if $\deg(A) \le \deg(B) - 2$ then $\sum_{j=1}^n A(z_j)/B'(z_j) = 0$. What happens if $\deg(A) = \deg(B) - 1$?

Since these identities are purely algebraic results, they must hold for polynomials over any algebraically closed field; but — as with invariance of the residue under coordinate change — a direct algebraic proof, though possible, is harder and less revealing.

9. Let E be the open right half-plane $\{z \in \mathbf{C} : \mathrm{Re}(z) > 0\}$, and $f : E \to \mathbf{C}$ a bounded analytic function. Suppose $f(x_k) = 0$ for some real numbers $x_k > 0$. Prove that

$$|f(1)| \le \left(\prod_{k=1}^{n} \frac{|1 - x_k|}{1 + x_k} \right) B$$

where $B = \sup_{z \in E} |f(z)|$, and find all f for which equality is attained. Deduce that if f is not identically zero but vanishes at $x_k > 0$ for each $k = 1, 2, 3, \ldots$ then $\sum_{k=1}^{\infty} 1/x_k < \infty$.

This underlies one approach to the proof of Müntz's theorem: if the topological span of $\{t^{x_k}\}_{k=1}^{\infty}$ is not dense in $\mathcal{C}(0,1)$ then its L^2 closure is a proper subspace of $L^2(0,1)$, and then there's a nonzero $\phi \in L^2(0,1)$ orthogonal to each t^{x_k}; then $f(x) := \int_0^1 \phi(t)\, t^x\, dt$ is holomorphic and bounded on E and vanishes at $x = x_k$ for each k, "etc.".

This problem set is due Friday, April 22, at the beginning of class.

[1] This is basically saying that $\mathbf{P}^1(\mathbf{C})$ is a 1-dimensional complex manifold.