Math 55b: Honors Real and Complex Analysis

Homework Assignment #10 (1 April 2011):
Integration in $\mathbf{R}^k$, and special functions

> **Similarly**, *adv.*: At least one line of the proof of this case is the same as before.[1]

Another foretaste of complex analysis:

1. Find all complex numbers s such that $\sum_{n=1}^{\infty} \mu(n) n^{-s}$ converges. Here μ is the Möbius function: if n is the product of $e \geq 0$ distinct primes then $\mu(n) = (-1)^e$; else $\mu(n) = 0$. Thus $\mu(n) = 1, -1, -1, 0, -1, 1, -1, 0, 0, 1$ for $n = 1, 2, 3, \ldots, 10$. (Hint: what is $1 / \sum_{n=1}^{\infty} \mu(n) n^{-s}$ for $\mathrm{Re}(s) > 1$?)

Some Rudin problems:

2.–3. Solve Problems 1 and 2 on page 288. Apropos #2, construct a bounded function $f : [0,1] \times [0,1] \to \mathbf{R}$ such that: for each x, the function $y \mapsto f(x,y)$ is Riemann integrable with $\int_0^1 f(x,y)\,dy = 0$ (from which it follows that $\int_0^1 (\int_0^1 f(x,y)\,dy)\,dx = 0$); but there exist y such that the function $y \mapsto f(x,y)$ is not Riemann integrable (whence $\int_0^1 (\int_0^1 f(x,y)\,dx)\,dy$ doesn't even make sense).

4.–8. Solve Problems 9 through 13 on pages 290–291. Generalize #13 to the integral of $\prod_{i=1}^{k} x_i^{r_i}$ over the set of $(x_1, \ldots, x_k)$ with each $x_i \geq 0$ and $\sum_{i=1}^{k} x_i^{s_i} = 1$. The r_i, s_i can be any real numbers with $r_i > -1$ and $s_i > 0$. (The resulting formula is due to Dirichlet.) In particular, determine the volume of the unit ball in $\mathbf{R}^k$ as a function of k; check that your answer agrees with the known cases $k = 1, 2, 3$. Note what happens to this volume as $k \to \infty$!

9. Let $Q : \mathbf{R}^n \to \mathbf{R}$ be a positive-definite form. Show that the integral of $e^{-Q(x)}$ over $x \in \mathbf{R}^n$ converges, and evaluate this integral. For $y \in \mathbf{R}^n$ determine the integral of $\exp(-Q(x) + i\langle x, y\rangle)$ over $x \in \mathbf{R}^n$, where $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$ is the usual inner product.

10. Let n be a positive integer, V the vector space of symmetric $n \times n$ matrices, and $E \subset V$ the open set of positive-definite matrices. Prove that the function $A \mapsto 1/\det(A)$ on E is logarithmically convex.

Problems 2–10 will be due Friday, April 8, at the beginning of class.

[1] *Definitions of Terms Commonly Used in Higher Math*, R. Glover et al. Note that this does not define an equivalence relation.