Math 55b: Honors Advanced Calculus and Linear Algebra

The Riemann-Stieltjes integral of a vector-valued function

At the end of Chapter 6 Rudin defines the integrals of functions from $[a, b]$ to $\mathbf{R}^k$ one coordinate at a time. This approach is problematic in infinite-dimensional vector spaces... Fortunately it is not too hard to adapt our definitions and results concerning Riemann-Stieltjes integration to functions from $[a, b]$ to an arbitrary normed vector space, as long as that space is complete — a condition we must impose so that the limiting process implicit in $\int$ will converge under reasonable hypothesis. We shall need to pay more attention to "Riemann sums", which Rudin is able to relegate to a fraction of Thm. 6.7 when treating integrals of real-valued functions.

Let V, then, be a *complete* normed vector space. Fix an interval $[a, b] \subset \mathbf{R}$, an increasing function $\alpha : [a, b] \to \mathbf{R}$, and a bounded function $f : [a, b] \to V$. For each partition $P : a = x_0 < x_1 < \cdots < x_n = b$ of $[a, b]$, and any choice of $t_i \in [x_{i-1}, x_i]$, we call

$$R(P, \vec{t}) := \sum_{i=1}^{n} \big(\alpha(x_i) - \alpha(x_{i-1})\big) f(t_i)$$

a *Riemann sum* for $\int_a^b f(x)\, d\alpha(x)$. (We suppress f, α from the notation $R(P, \vec{t})$ because f, α are fixed for this discussion.) When $V = \mathbf{R}$ we can estimate all the Riemann sums above and below by $U(P)$ and $L(P)$, and try to make the difference between these upper and lower bounds arbitrarily small by choosing a sufficiently fine partition P. For an arbitrary V there is no "above" and "below", and thus no $U(P)$ and $L(P)$; but we can still formulate a generalization of $U(P) - L(P)$ that bounds how much ambiguity the choice of $\vec{t}$ entails. For any c, d with $a \le c < d \le b$, let

$$E(c, d) := \sup_{t, t' \in [c,d]} |f(t) - f(t')|;$$

note that the sup exists because f is bounded. Then, for any partition P define

$$\Delta(P) := \sum_{i=1}^{n} \big(\alpha(x_i) - \alpha(x_{i-1})\big)\, E(x_{i-1}, x_i).$$

Then

$$\left| R(P, \vec{t}\,') - R(P, \vec{t}) \right| \le \Delta(P)$$

for any choices of $t_i, t_i' \in [x_{i-1}, x_i]$.

When $V = \mathbf{R}$ this $\Delta(P)$ coincides with $U(P) - L(P)$ (why?). For this to work in the general setting, we must verify that if P^* refines P then $\Delta(P^*) \le \Delta(P)$. By induction we need only check that if $a \le x < y < z \le b$ then

$$\big(\alpha(y) - \alpha(x)\big)E(x, y) + \big(\alpha(z) - \alpha(y)\big)E(y, z) \le \big(\alpha(z) - \alpha(x)\big)E(x, z).$$

But this is clear from $E(x, y) \leq E(x, z)$ and $E(y, z) \leq E(x, z)$. Furthermore, we need the following: if P^* refines P, any Riemann sums for P^* and P differ by at most $\Delta(P)$. Again it is enough to prove this for a one-point refinement of $[x, z]$ to $[x, y] \cup [y, z]$, when we claim

$$\begin{aligned}\big|\big(\alpha(y) - \alpha(x)\big) f(t_1) + \big(\alpha(z) - \alpha(y)\big) f(t_2) - \big(\alpha(z) - \alpha(x)\big) f(t)\big| \\ \leq \big(\alpha(z) - \alpha(x)\big) E(x, z).\end{aligned}$$

To prove this, we write the LHS as

$$\big|\big(\alpha(y) - \alpha(x)\big)\big(f(t_1) - f(t)\big) + \big(\alpha(z) - \alpha(y)\big)\big(f(t_2) - f(t)\big)\big|$$

and use $|f(t_i) - f(t)| \leq E(x, z)$.

We can now recast Thm. 6.6 as a definition for Riemann-Stiltjes integrability of f: we say that f is *integrable* with respect to $d\alpha$ if for each ϵ there is a partition P such that $\Delta(P) < \epsilon$. [By Thm. 6.6, this is equivalent to Rudin's definition in the case $V = \mathbf{R}$.] Theorems 6.8 through 6.10 then generalize immediately to sufficient conditions for integrability of vector-valued functions.

But you'll notice that we have managed to define integrability without defining the integral! The definition as follows: if f is integrable with respect to $d\alpha$, its *integral* $\int_a^b f(x)\, d\alpha(x)$ is the unique $I \in V$ such that $|I - R(P, \vec{t})| \leq \Delta(P)$ for all Riemann sums $R(P, \vec{t})$. Of course this requires proof of existence and uniqueness. Uniqueness is easy: if two distinct I, I' worked, we'd get a contradiction by choosing P such that $\Delta(P) < \frac{1}{2}|I' - I|$. We next prove existence. First we construct I. Let $\epsilon_m \to 0$ and choose P_m such that $\Delta(P_m) < \epsilon_m$. Without loss of generality we may assume P_m refines $P_{m'}$ for each $m' < m$, by replacing P_m by a common refinement of $P_1, \ldots, P_m$ (this cannot increase $\Delta(P_m)$). Choose for each m an arbitrary Riemann sum $R_m := R(P_m, \vec{t}(m))$. These constitute a Cauchy sequence in V: if $m' < m$ then $|R_{m'} - R_m| \leq \Delta(P_{m'}) < \epsilon_{m'}$. We now at last use the completeness of V to conclude that $\{R_m\}$ converges. Let I be its limit. This of course is essentially the way we obtain the integral of a real-valued function. We claim that I satisfies our requirements for an integral even in our vector-valued setting. Consider any Riemann sum $R = R(P, \vec{t})$. For each m, let R_m^* be a Riemann sum for a common refinement of P_m and P. Then $|R_m^* - R| \leq \Delta(P)$ and $|R_m^* - R_m| \leq \Delta(P_m) < \epsilon_m$. Thus $|R_m - R| < \Delta(P) + \epsilon_m$. Letting $m \to 0$ we obtain $|I - R| \leq \Delta(P)$ as desired, Q.E.D.