Math 55b: Honors Advanced Calculus and Linear Algebra

Practice Problems (May 2, 2003)

As was true last term, these problems are not required homework, but are meant to help you review the material for the final exam. Good luck!

Fourier/Hilbert stuff:

1. (Baire's theorem) Prove that if a complete metric space X is the countable union $\cup_{n=1}^{\infty} S_n$ with each $S_n \subset X$ closed then at least one S_n contains a nonempty open set. [This involves the same argument we used to show that a weakly convergent sequence of operators is bounded.]

2. Prove that the roots of consecutive orthogonal polynomials are interlaced, i.e., that if u_j are orthogonal polynomials with respect to some weight function $w(x)$ on (a, b), and the roots of u_n and u_{n+1} are $x_1 < x_2 < x_3 < \cdots < x_n$ and $x'_1 < x'_2 < \cdots < x'_{n+1}$ respectively, then

$$x'_1 < x_1 < x'_2 < x_2 < x'_3 < \cdots < x'_n < x_n < x'_{n+1}.$$

   [It suffices to prove that, for each k, $u_{n+1}(x_k)$ is a nonzero real number of sign $(-1)^{n+1-k}$ (why?). Use induction and "Theorem 40.9".] Show more generally that if $m > n$ then there is at least one zero of u_m between each pair of zeros of u_n. Check directly that this holds for the Tchebychev polynomials using the explicit values of x_k and x'_k. What can you say about the zeros of a linear combination $u_n + cu_{n-1}$?

3. Give a proof using convolutions of Parseval for functions on $\mathbf{T}$ and/or $\mathbf{Z}/N\mathbf{Z}$.

The matrix exponential and logarithm:

4. Let $\mathcal{M}$ be the n^2-dimensional vector space of $n \times n$ matrices. Define a map $\exp : \mathcal{M} \to \mathcal{M}$ (the matrix exponential) by

$$\exp(A) = \sum_{n=0}^{\infty} A^n/n! = \mathbf{1} + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots$$

   [Here $\mathbf{1}$ is the identity matrix in $\mathcal{M}$. To show that the sum converges absolutely and uniformly in compact subsets of $\mathcal{M}$, note that each of the n^2 coordinates is dominated by $\sum_{n=0}^{\infty} \|A\|^n/n!$ where $\|A\| = \sup_{|\mathbf{x}|\leq 1} |A\mathbf{x}|$ is the norm of A.] Prove that exp is continuously differentiable near the origin $\mathbf{0}$ of $\mathcal{M}$, and find its differential at $\mathbf{0}$. Conclude that exp has a local inverse near $\mathbf{1} = \exp \mathbf{0}$. Note that exp is not globally invertible: give examples showing that it is neither surjective (prove that $\exp(A)\exp(-A) = \mathbf{1}$ for all $A \in \mathcal{M}$) nor injective (consider $A = t\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ for suitable t). What is $\det(\exp A)$? If $B \in \mathrm{GL}_n(\mathbf{R})$, must there exist $A \in \mathcal{M}$ such that $B = \exp A$?

It should not surprise you that the inverse function of exp is called the "matrix logarithm" and is given near $\mathbf{1}$ by the power series $\log(\mathbf{1} + E) = E - E^2/2 + E^3/3 - + \cdots$ for E in a neighborhood of $\mathbf{0}$.

Some miscellaneous calculus problems follow; there are plenty more in Rudin to choose from:

5. (Euler's original evaluation of $\zeta(2)$, etc.) Recall from an earlier problem set the infinite product

$$\sin x = x \prod_{n=1}^{\infty} \left[1 - \left(\frac{x}{n\pi}\right)^2\right]$$

for x in some neighborhood of 0. Taking logarithms of both sides yields

$$\log \frac{\sin x}{x} = \sum_{n=1}^{\infty} \log \left[1 - \left(\frac{x}{n\pi}\right)^2\right],$$

Now carefully expand the logarithms on the right-hand side in power series about $x = 0$, and compare the leading (x^2) coefficient with that of

$$\log(\sin x/x) = \log(1 - x^2/6 + x^4/120 - + \cdots)$$

to recover Euler's identity. What do the further coefficients of the Taylor expansion tell you?

6. Suppose f is a harmonic function on the neighborhood of a closed ball $\bar{B}_r(x)$ in $\mathbf{R}^n$. Prove that $f(x)$ is the average of $f(y)$ over $y \in \bar{B}_r(x)$, and also the average of $f(y)$ over the sphere $|y - x| = r$. Prove that conversely if f is a $\mathcal{C}^2$ function from $\bar{B}_r(x)$ to $\mathbf{R}$ whose average over $B_s(x)$ equals $f(x)$ for each positive $s < r$ then $\Delta f(x) = 0$. (This generalizes the familiar characterization $f(x+r)+f(x-r) = 2f(x)$ for affine-linear functions of one variable $f : [x - r, x + r] \to \mathbf{R}$. An important application is that a gravitational or electrostatic field cannot have points of stable equilibrium!)