Math 55b: Honors Advanced Calculus and Linear Algebra

Parseval's identity for Fourier transforms

Parseval's identity for Fourier transforms can be stated as follows:

Theorem. *There exists a linear operator $f \mapsto \hat{f}$ on $L_2([0,1])$ that coincides with the Fourier transform on bounded L_1 functions and satisfies the identity*

$$(\hat{f}, \hat{g}) = 2\pi (f, g)$$

for all $f, g \in L_2([0,1])$.

This identity, or its special case $|\hat{f}|^2 = 2\pi |f|^2$, is known as *Parseval's identity.* The two forms are equivalent, since applying the special case to $f \pm g$ and $f \pm ig$ recovers the general case. (We have seen this "polarization" trick last term; remember "quarter squares"?) Yet another equivalent form is: the operator $f \mapsto (2\pi)^{-1/2}\hat{f}$ is unitary. This is actually not quite equivalent: a unitary operator must not only preserve the norm but be bijective. On an infinite-dimensional Hilbert space there are operators that are norm-preserving but not surjective, such as $(a_1, a_2, a_3, \ldots) \mapsto (0, a_1, a_2, \ldots)$ on l_2. But the map $f \mapsto (2\pi)^{-1/2}\hat{f}$ is known to be bijective, because its fourth iterate is the identity by the inversion formula.

Since both sides of Parseval's identity are continuous in f, it is enough to prove the identity for f in a dense subset of $L_2([0,1])$. We know that the continuous functions supported on finite intervals constitute such a subset. Let f, then, be a continuous function of compact support. Write (f, f) as a convolution:

$$(f, f) = \int_{-\infty}^{\infty} f(t)\overline{f(t)}\, dt = \int_{-\infty}^{\infty} f(t) f_1(-t)\, dt = (f * f_1)(0),$$

where $f_1(t) := \overline{f(-t)}$ is also a continuous function of compact support. Thus the same is true of $g := f * f_1$. Therefore $\hat{g} = \hat{f}\hat{f}_1$. We readily find that $\hat{f}_1(\zeta) = \overline{\hat{f}(\zeta)}$, and conclude that $\hat{g}(\zeta) = |\hat{f}(\zeta)|^2$ for all $\zeta \in \mathbf{R}$. Since g is continuous we may apply Féjer's theorem for Fourier transforms to find

$$g(0) = \frac{1}{2\pi} \lim_{R \to \infty} \int_{-R}^{R} \left(1 - \frac{|\zeta|}{R}\right) |\hat{f}(\zeta)|^2\, d\zeta.$$

But $|\hat{f}(\zeta)|^2 \geq 0$ for all ζ, so the limit exists if and only if $\int_{-\infty}^{\infty} |\hat{f}(\zeta)|^2\, d\zeta$ converges in which case it equals that integral. Thus

$$(f, f) = g(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\zeta)|^2\, d\zeta = \frac{1}{2\pi} (\hat{f}, \hat{f})$$

as desired, and we are done.

(The same approach can also prove $(\hat{f}, \hat{g}) = 2\pi(f, g)$ directly — once it is known that $\hat{f}, \hat{g}$ are in L_2 so that $\hat{f}\hat{g}$ is L_1 by Cauchy-Schwarz.)

As an application we prove:

Theorem. *For any L_1 function $f : \mathbf{R} \to \mathbf{C}$, its Fourier transform $\hat{f}(\zeta)$ approaches 0 as $|\zeta| \to \infty$.*

Recall that we have already shown that $\hat{f}$ is bounded by $\int_{-\infty}^{\infty} |f(t)|\, dt$. Thus again it is enough to prove our claim for f in a dense subset S of $L_1(\mathbf{R})$. Indeed for each ϵ we may then approximate f by some $f_1 \in S$ to within $\epsilon/2$ in the L_1 norm; then $|\hat{f}(\zeta) - \hat{f}_1(\zeta)| \leq \epsilon/2$ for all ζ, since $\hat{f} - \hat{f}_1$ is the Fourier transform of $f - f_1$. If it is known that $\hat{f}_1(\zeta) \to 0$ as $|\zeta| \to \infty$, then there exists R such that $|\hat{f}_1(\zeta)| < \epsilon/2$ for $|\zeta| > R$; thus $|\hat{f}(\zeta)| < \epsilon$ for $|\zeta| > R$. Since ϵ is arbitrary, it will follow that $f(\zeta) \to 0$ as $|\zeta| \to \infty$ as desired.

We may use the L_2 functions in $L_1(\mathbf{R})$ as the dense subset S, since the bounded, compactly supported functions are dense in $L_1(\mathbf{R})$ and a compactly supported integrable function is automatically square integrable. By Parseval, if f_1 is in this S then so is $\hat{f}_1$. But we know already (Lemma 46.3) that $\hat{f}_1$ is uniformly continuous. Well, a uniformly continuous L_2 function on $\mathbf{R}$ must tend to 0 as $|\zeta| \to \infty$. Indeed if it didn't we would have an $\epsilon > 0$ and infinitely many ζ_n such that $|\hat{f}_1(\zeta_n)| \geq \epsilon$ and $|\zeta_m - \zeta_n| > 1$ for $m \neq n$. Since $\hat{f}_1$ is uniformly continuous, there exists $\delta > 0$ such that $|\hat{f}_1(\zeta) - \hat{f}_1(\zeta')| < \epsilon/2$ if $|\zeta - \zeta'| < \delta$. We may assume that $\delta < 1/2$. Then the δ-balls about the ζ_n are disjoint and we have

$$\infty > \int_{-\infty}^{\infty} |\hat{f}_1(\zeta)|^2\, d\zeta \geq \sum_{n=1}^{\infty} \int_{\zeta_n-\delta}^{\zeta_n+\delta} |\hat{f}_1(\zeta)|^2\, d\zeta \geq \sum_{n=1}^{\infty} \int_{\zeta_n-\delta}^{\zeta_n+\delta} (\epsilon/2)^2 d\zeta = \sum_{n=1}^{\infty} \frac{\delta\epsilon^2}{2},$$

which is impossible. Our theorem is thus proved.

We could have used other subsets S, such as the functions of bounded variation, for which we know $\hat{f}_1(\zeta) \ll 1/|\zeta|$. We use the L_2 functions because this approach generalizes most readily to the Fourier transform on an arbitrary locally compact abelian group.