**Math 55b: Honors Advanced Calculus and Linear Algebra**

Homework Assignment #9 (4 April 2003):
Fourier Series I

Beware the Ides of April! — *the Infernal Revenue Service*[1]

A few simple observations on Fourier coefficients, and more about "summability methods":

1. i) If $g : \mathbf{T} \to \mathbf{C}$ is integrable then so is its complex conjugate $\bar{g}$; determine the Fourier coefficients of $\bar{g}$ in terms of those of $g$, and use this to show that if $g$ is actually real-valued then its $n$-th coefficient $\hat{g}_n$ is the complex conjugate of $\hat{g}_{-n}$. Find a similar result for even or odd functions $g$ [that is, functions $g : \mathbf{T} \to \mathbf{C}$ satisfying the identity $g(-t) = g(t)$ or $g(-t) = -g(t)$].
   ii) Prove conversely that if $\overline{\hat{g}_n} = \hat{g}_{-n}$ for all $n \in \mathbf{Z}$ then $g(t) \in \mathbf{R}$ for all $t \in \mathbf{T}$ at which $g$ is continuous. Likewise obtain a converse for the results of (i) for even and odd functions.

2. Let $f : \mathbf{T} \to \mathbf{C}$ be an integrable function, and define for $t \in \mathbf{T}$ and $|z| < 1$
$$\phi_t(z) := \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int} z^{|n|}.$$
   Prove that if $f$ is continuous at $t \in \mathbf{T}$ then $\phi_t(z)$ approaches $f(t)$ as $z \to 1$, and that if $f$ is continuous at all $t \in \mathbf{T}$ then the limit $\lim_{z\to 1} \phi_t(z) = f(t)$ is uniform in $t$. [Mimic Fejér's proof, evaluating $\sum_n z^{|n|} e^{ins}$ as a sum of two geometric series.]

3. A series $\sum_{n=-\infty}^{\infty} a_n$ is said to be *Abel summable* if $\phi(z) := \sum_n a_n z^{|n|}$ converges for all $z \in [0, 1)$ and $\lim_{z\to 1} \phi(z)$ exists, in which case that limit is called the *Abel sum* of the series. (Thus Problem 2 showed that the Fourier series of $f$ is Abel summable to $f$ wherever $f$ is continuous.)
   i) Prove that if some series $\sum_{n=-\infty}^{\infty} a_n$ is Cesàro summable then it is also Abel summable and the Cesàro and Abel sums are equal. [Of course this result together with Fejér's theorem gives another proof of Problem 2.]
   ii) Find real numbers $a_n$ such that the series $\sum_{n=-\infty}^{\infty} a_n$ is Abel summable but not Cesàro summable. [Thus Abel summability is a strictly weaker condition than Cesàro summability.]

More about Weyl equidistribution:

4. Let $t_1, t_2, t_3, \ldots$ be a sequence of real numbers mod $2\pi$ (= elements of $\mathbf{T}$). Assume that
$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} e^{ist_r} = 0$$

[1]Actually the Ides of April, and of every other month except March, May, July and October, falls on the 13th, not the 15th of the month. Too bad.

for all nonzero $s$, except that for $s = \pm 1$ we instead have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} e^{\pm i t_r} = 1/2.$$

What can you conclude about the limits

$$\lim_{n\to\infty} \frac{1}{n} \#\{r \in \mathbf{Z} \,|\, 1 \le r \le n,\ 2\pi a \le t_r \le 2\pi b\}$$

$(0 \le a \le b \le 1)$ and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f(t_r)$$

$(f : \mathbf{T} \to \mathbf{C}$ continuous)? Generalize.

5. Let $\gamma \in \mathbf{R}$ be any irrational number. Prove that the sequence $t_r = r^2\gamma$ is equidistributed mod 1. [Again we need to estimate $\sum_{r=1}^{n} e^{2\pi i s t_r}$ for nonzero integers $s$. To simplify the notation let $z$ be the complex number $e^{2\pi i s \gamma}$ of absolute value 1, so we need to show that $(\sum_{r=1}^{n} z^{r^2})/n \to 0$ as $n \to \infty$. The trick is to fix some integer $h$ and rewrite our sum for $n > h$ as

$$\sum_{r=1}^{n} z^{r^2} = \frac{1}{h} \sum_{m=1}^{n} \left( \sum_{r=m}^{m+h-1} z^{r^2} \right) + E$$

with the error $E$ bounded by $h$. By Cauchy-Schwarz this is at most

$$\frac{1}{h} \left( n \sum_{m=1}^{n} \left| \sum_{r=m}^{m+h-1} z^{r^2} \right|^2 \right)^{1/2} + E.$$

Now show that, since $z$ is not a root of unity (why?), the sum over $m$ is at most $hn + Ch^2$, with $C$ depending on $\gamma$ and $h$ but not on $n$. Thus $|\sum_{r=1}^{n} z^{r^2}|/n < h^{-1/2} + o(1)$ as $n \to \infty$. Finally let $h \to \infty$.]

With some more courage and perseverance one can repeat this argument to show inductively that for any polynomial $P(x)$ the sequence $\{P(r)\}_{r=1}^{\infty}$ is equidistributed mod 1, provided at least one nonconstant coefficient of $P$ is irrational. This was Weyl's original application of his equidistribution theorem in 1914.

Relating the Fourier expansion of a differentiable function to that of its derivative(s):

6. i) If $f : \mathbf{T} \to \mathbf{C}$ is a $\mathcal{C}^1$ function, express the Fourier coefficients of its derivative in terms of the coefficients $\hat{f}_n$ of $f$.
   ii) Prove that $f$ is $\mathcal{C}^\infty$ if and only if $|n|^k \hat{f}_n \to 0$ as $n \to \pm\infty$ for all $k$.
   iii) Show that the function $F(t) = \sum_{n=1}^{\infty} n^{-5/2} \sin(nt)$ does <u>not</u> have a continuous second derivative on $\mathbf{T}$.

7. (Wirtinger's Inequality) Let $f : [0, \pi] \to \mathbf{R}$ be a $\mathcal{C}^1$ function with right and left derivatives at 0 and $\pi$ respectively. Assume that $f(0) = f(\pi) = 0$. Prove that $\int_0^\pi f(x)^2\,dx \le \int_0^\pi f'(x)^2\,dx$ with equality if and only if $f(x) = c\sin(x)$ for some $c \in \mathbf{R}$. [Extend $f$ to a $\mathcal{C}^1$ function on $\mathbf{T}$ by $f(-x) = -f(x)$. Challenge: can you prove this inequality without using Fourier analysis? Can you relax the assumption that $f$ is differentiable at the endpoints?]

This problem set is due Friday, April 11 in class.