Math 55b: Honors Advanced Calculus and Linear Algebra

Homework Assignment #8 (31 March 2003):
Life in Hilbert space

It is a mathematical fact that the casting of this pebble from my hand alters the centre of gravity of the universe.
— Thomas Carlyle (1795–1881), who apparently knew his John Donne ("if a clod be washed away by the sea, Europe is the less"[1]) but was not as clear on Newton's Laws...

1. (More about separability) Let X be a metric space, and S any subspace. Prove that if X is separable than so is S. Prove that the following are equivalent:
   i) S is separable;
   ii) For each $\epsilon > 0$ there is a countable set of ϵ-neighborhoods in X whose union contains S;
   iii) For each $\epsilon > 0$ there is a countable set of ϵ-neighborhoods in S whose union contains S.

2. i) Show directly (i.e., without Weierstrass approximation or Fourier analysis) that $\mathcal{C}([0,1])$ is separable. (As noted in class, the separability of $L_2([0,1])$ follows.)
   ii) The space $L_2(\mathbf{R})$ is defined as the completion of the space of continuous functions on $\mathbf{R}$ with compact support, relative to the usual inner product $(f,g) := \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx$. [Since f, g have comapct support, this is actually an integral over a bounded interval, so it makes sense.] Prove that $L_2(\mathbf{R})$ is separable, and thus isometric with l_2. Can you find an explicit ontb for $L_2(\mathbf{R})$?

3. (Compact subsets of l_2):
   i) Let $r_1, r_2, r_3, \ldots$ be a sequence of positive real numbers. Prove that the "box"
   $$\{x \in l_2 : |x_1| \le r_1, |x_2| \le r_2, \ldots\}$$
   in l_2 is compact if and only if it is bounded, that is, <u>iff</u> $\sum_{i=1}^{\infty} r_i^2 < \infty$.
   ii) Prove that the "ellipsoid" $\{x \in l_2 : \sum_{i=1}^{\infty} |x_i/r_i|^2 \le 1\}$ is compact <u>iff</u> $r_i \to 0$ as $i \to \infty$.

A linear map $L : V \to W$ between complete normed spaces V, W is said to be **compact** if the closure in W of the image of the closed unit ball $\bar{N}_1(0)$ in V is a compact subset of W. Clearly a compact map must be bounded (why?). Thus the last part of Problem 3 shows that the linear map $l_2 \to l_2$ defined by $(x_1, x_2, x_3, \ldots) \mapsto (r_1x_1, r_2x_2, r_3x_3, \ldots)$ is compact <u>iff</u> $r_i \to 0$.[2] Note that this linear map is self-adjoint, and the unit vectors

[1] From Donne's Meditation XVII, which more famously is the source of "no man is an island" and "never send to know for whom the bell tools; it tolls for thee". The Carlyle quote is from *Sartor Resartus*, according to several sources on the Web. No, this has no specific connection with Hilbert space, separability, or Fourier analysis.

[2] Here $\{r_i\}$ is any bounded sequence of scalars, which need not be positive reals.

constitute an orthonormal topological basis of eigenvectors. As was true in the finite dimensional case, there is a **spectral theorem** for compact self-adjoint linear transformations L of an infinite-dimensional separable Hilbert space $\mathcal{H}$, which states that any such L has an ontb of eigenvectors and thus is equivalent to the linear map above under a suitable identification of $\mathcal{H}$ with l_2 (i.e. using those eigenvectors as the unit vectors of l_2). The proof is outlined in the next two problems, under the slightly stronger hypothesis that $L(\bar{N}_1(0))$ [rather than $\overline{L(\bar{N}_1(0))}$] is compact. You may assume that $\mathcal{H}$ is a real Hilbert space; the complex case is essentially the same, but with a few extra wrinkles.

4. Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space, and $L : \mathcal{H} \to \mathcal{H}$ a self-adjoint linear map such that $L(\bar{N}_1(0))$ is compact. Since $K = \{L(x) : x \in \mathcal{H}, |x| \leq 1\}$ is compact, it contains some vector $L(x_1)$ of maximal norm. Prove that (unless L is identically zero, in which case we're done already) $|x_1| = 1$ and x_1 is an eigenvector of L^2. Show also that if x is any vector of $\mathcal{H}$ orthogonal to x_1 then $L^2(x)$ is also orthogonal to x_1, i.e. that L^2 restricts to a linear map from the orthogonal complement of $\langle x_1 \rangle$ to itself.

5. Note that this restriction is itself compact and self-adjoint. Using the result of the previous problem, obtain orthonormal eigenvectors $x_2, x_3, \ldots$ of L^2, and use these to prove the spectral theorem for compact self-adjoint linear operators L on $\mathcal{H}$.

6. Show that for each $x \in [0, 1]$ the functional $f \mapsto \int_0^x f(t)\,dt$ on $\mathcal{C}([0, 1])$ extends continuously to a functional ϕ_x on $L_2([0, 1])$. What is the norm of ϕ_x? Show that for any $f \in L_2([0, 1])$ the map $Tf : x \mapsto \phi_{1-x}(f)$ is continuous. Prove that $f \mapsto Tf$ is a continuous linear transformation from $L_2([0, 1])$ to $\mathcal{C}([0, 1])$, and thus to $L_2([0, 1])$. Prove further that T, considered as a linear operator on $L_2([0, 1])$, is self-adjoint and compact. Find an ontb of eigenvectors of T.

This abbreviated problem set is due Friday, April 4 in class.