Math 55b: Honors Advanced Calculus and Linear Algebra

Homework Assignment #7 (π Day (March 14), 2003):
Differential forms, chains, integration, and more exterior algebra

$\mathfrak{so}_6\mathbf{C} \cong \mathfrak{sl}_4\mathbf{C}$
— Fulton and Harris, *Representation Theory: a First Course* (Springer, 1991), page 282 and elsewhere. This is one version of the "ultimate explanation" of the identification of the lines in a four-dimensional vector space with the points of a quadric in a five-dimensional projective space over the same field.

1. Let $E \subset \mathbf{R}^2$ be the punctured plane $\mathbf{R}^2 - \{\mathbf{0}\}$. Recall that we have constructed a closed but not exact 1-form $d\theta$ on E. Show that any closed 1-form ω on E can be written uniquely as $\phi + c\, d\theta$ where ϕ is an exact 1-form and $c \in \mathbf{R}$. [Thus "the first (deRham) cohomology $H^1(E, \mathbf{R})$ is one-dimensional", since it is generated by the class of $d\theta$.] Give a formula for c in terms of ω.

2. Prove that every closed affine 1-chain in a *convex* set $E \subseteq \mathbf{R}^n$ is the boundary of some affine 2-chain in E.

3. Let $E \subseteq \mathbf{R}^n$ be a convex set, and $\gamma : [a, b] \to E$ any $\mathcal{C}^m$ curve. Construct a $\mathcal{C}^m$ 2-chain in E whose boundary is the difference between γ and the affine 1-simplex $[\gamma(a), \gamma(b)]$. [Hint: rather than working directly with 2-simplices it will be easier to use a 2-cell and then apply Exercise 17.] Conclude from this and the previous problem that *every closed $\mathcal{C}^m$ 1-chain in a convex set is a boundary of a $\mathcal{C}^m$ 2-chain in the same convex set.*

Since we have shown that $\partial^2 = 0$, this means that a 1-chain in a convex set is closed if and only if it is a boundary. The same is true for k-chains; the proof uses the same basic ideas, but requires rather more bookkeeping. For both your and Andrei's sake I'll leave the details to a future course in algebraic or differential topology.

The next two problems from Rudin construct and investigate closed but not exact $(n-1)$-forms on $\mathbf{R}^n - \{0\}$, generalizing our form $d\theta$ on the punctured plane. These are a key ingredient in the proof of the Brouwer fixed-point theorem and related results (such as the "ham sandwich theorem" and its generalization to $\mathbf{R}^n$), at least for sufficiently differentiable functions.

4.–5. Solve problems 22 and 23 in the text (pages 294–296).

6. Solve problem 29 (page 297).

Finally, a sorbet of exterior algebra:

7. i) Let V be a finite-dimensional real inner product space. Prove that there is a unique inner product on $\wedge^d V$ such that $\langle (v_1 \wedge \cdots \wedge v_d), (v_1' \wedge \cdots \wedge v_d') \rangle = \det(\langle v_i, v_j' \rangle)_{i,j=1}^d$ for any $v_1, \ldots, v_d, v_1', \ldots, v_d' \in V$.

   ii) Now let V have dimension 4 and $W = \wedge^2 V$. Fix a generator δ of $\wedge^4 V$ such that $\langle \delta, \delta \rangle = 1$. (We may take $\delta = e_1 \wedge e_2 \wedge e_3 \wedge e_4$ where $e_1, \ldots, e_4$ is an orthonormal basis for V.) We then have a bilinear pairing $(\cdot, \cdot) : W \times W \to \mathbf{R}$ defined by $w \wedge w' = (w, w')\delta$. In the last problem set we showed in effect that this pairing is nondegenerate, and thus identifies W with W^*. But now that V has an inner product structure we have another such pairing, $\langle \cdot, \cdot \rangle$, and thus another identification of W with its dual. Composing one of these two identifications with the other's inverse yields a map $\iota : W \to W$ characterized by $\langle w, w' \rangle = (\iota w, w')$ for all $w, w' \in W$. Prove that ι is an involution each of whose eigenspaces $W_\pm := \{w \in W : \iota w = \pm w\}$ has dimension 3.

   iii) For any $v \in V$, show that $\{v \wedge v' | v \in V\}$ and $\{v_1 \wedge v_2 | v_i \in V, \langle v, v_i \rangle = 0\}$ are isotropic subspaces of W of dimension 3, the maximum for an isotropic space for a pairing on a 6-dimensional space. Show that any 3-dimensional isotropic subspace is of one of these two forms, and that ι takes maximal isotropics of one kind to the other.

   iv) Recall that any linear transformation T of V induces a linear transformation $\wedge^2 T$ of W [by $(\wedge^2 T)(v \wedge v') = (Tv) \wedge (Tv')$]. Show that if T is orthogonal then $\wedge^2 T$ takes W_+ to either W_+ or W_-. Which one is it?

This problem set is due Friday, March 21 in class.