Math 55b: Honors Advanced Calculus and Linear Algebra

Homework Assignment #6 (7 March 2003):
Interlude on convexity; introduction to differential forms

If you see a good move, don't play it: look for a better one!
Edward Lasker (1868–1941), mathematician and World Chess Champion.

In other words, If you see a complicated way to solve a problem, stop — before carrying it through to completion, check whether a simpler approach is available. In the long run this will save you time and reduce the probability of error.

First, some basic facts about convex functions. Recall that a subset E of a real vector space V is said to be *convex* if $x, y \in E \Rightarrow px + qy \in E$ for all $p, q \in [0, 1]$ such that $p + q = 1$. If E is convex, an (upward) *convex function* on E is a function $f : E \to \mathbf{R}$ such that $f(px + qy) \leq pf(x) + qf(y)$ for all $x, y \in E$, $p, q \in [0, 1]$ with $p+q = 1$; equivalently, f is convex if $\{(x, t) \in V \oplus \mathbf{R} : t > f(x)\}$ is a convex subset of $V \oplus \mathbf{R}$.

1. i) Show that any convex function on a convex open set in $\mathbf{R}^k$ is continuous.
   ii) Let U be a convex open set in $\mathbf{R}^k$, and fix $B \in (0, \infty)$ and a compact subset $K \subset U$. Let $\mathcal{C}$ be the set of all convex functions $f : U \to [-B, B]$. Prove that the restriction of $\mathcal{C}$ to the space of continuous functions on K is equicontinuous.

2. [Jensen's inequalities] Let f be a convex function on a convex set E in some real vector space.
   i) If $x_i \in E$, $p_i \geq 0$, and $\sum_{i=1}^n p_i = 1$, prove that $x := \sum_{i=1}^n p_i x_i$ is in E and $f(x) \leq \sum_{i=1}^n p_i f(x_i)$. (This contains many classical inequalities as special cases; e.g., the inequality on the arithmetic and geometric means is obtained by taking $E = (0, \infty)$, $f(x) = -\log x$, and $p_i = 1/n$.)
   ii) If $\phi : [a, b] \to E$ is a continuous function and $\alpha : [a, b] \to \mathbf{R}$ is an increasing function such that $\alpha(b) - \alpha(a) = 1$, prove that $x := \int_a^b \phi(t)\, d\alpha(t)$ is in E and $f(x) \leq \int_a^b f(\phi(t))\, d\alpha(t)$.

3. The logarithmic convexity of $\Gamma(x)$, or more generally of any function of the form $f(x) = \int (\alpha(t))^x \beta(t)\, dt$, can be interpreted as the nonnegativity of the determinant of a symmetric 2×2 matrix. Generalize this to larger determinants. For instance, prove that for any positive reals $a_1, \ldots, a_n$ the determinant of the $n \times n$ matrix with entires $\Gamma(a_i + a_j)$ is nonnegative, as is the determinant with entries $(a_i + a_j)^{-k}$ for any $k > 0$. [Hint for this last part: what is $\int_0^\infty t^{x-1} e^{-ct}\, dt$?]

The next problem reviews exterior algebra:

4. Let V be a vector space over some field k, and let U be a subspace of finite dimension d. For any basis $v_1, \ldots, v_d$ of U, consider $\omega := v_1 \wedge v_2 \wedge \cdots \wedge v_d \in \bigwedge^d V$.

   i) Show that $\omega \neq 0$, and that any other choice of basis would yield a nonzero scalar multiple of ω. Moreover, if $U' \subset V$ is a subspace of dimension d for which the same procedure yields a scalar multiple of ω then $U' = U$.

   ii) Now suppose V is of dimension 4 over a field k not of characteristic 2, and let W be the k-vector space $\bigwedge^2 V$ of dimension 6. Fix a generator δ of the one-dimensional space $\bigwedge^4 V$. Then for each $\omega \in W$ we have $\omega \wedge \omega = Q(\omega)\delta$ for some $Q(\omega) \in k$. Prove that $\omega \mapsto Q(\omega)$ is a nondegenerate quadratic form, and that $Q(\omega) = 0$ if and only if $\omega = v_1 \wedge v_2$ for some $v_1, v_2 \in V$. (This, combined with (i), identifies the lines in V with a quadric in the 5-dimensional projective space $(W - \mathbf{0})/k^*$.) If $k = \mathbf{R}$, what is the signature of Q?

Simplices, boxes, etc.:

5–6. Solve exercises 16–19 on pages 291–2. (Cf. "Remark 10.41" on page 280.)

Finally, a more-or-less explicit approach to the simplest case of "closed on convex is exact", and an application to a fundamental fact about complex analysis:

7. Solve exercises 24, 25 on page 296. [Such arguments are ubiquitous in differential geometry and algebraic topology.]

8. [Conjugate harmonic functions] In problem 5 of the 4th set we obtained the "Cauchy-Riemann equations" for the real and complex parts u, v of a differentiable function $u + iv$ on an open subset E of $\mathbf{C}$.

   i) Rewrite these partial differential equations as formulas for the exact differentials df, dg in terms of g, f respectively. Use this to show that, if E is convex, for any harmonic function $f \in \mathbf{C}^2(E)$ there exists g, unique up to additive constants, such that f, g satisfy the Cauchy-Riemann equations; and conversely that given a harmonic g there exists f. (Such f, g are called "conjugate harmonic functions" on E.)

   ii) Now let E be the non-convex set $\mathbf{C} - \{0\}$. Give an example of a harmonic $\mathbf{C}^\infty$ function $f : E \to \mathbf{R}$ that has no conjugate harmonic function $g : E \to \mathbf{R}$.

This problem set due π Day, March 14, at the beginning of class.