Math 55b: Honors Advanced Calculus and Linear Algebra

Homework Assignment #5 (28 February 2003):
Integration in $\mathbf{R}^k$, and special functions

Similarly, *adv.*: At least one line of the proof of this case is the same as before.[1]

1.–2. Solve Problems 1 and 2 on page 288. Apropos #2, construct a bounded function $f : [0,1] \times [0,1] \to \mathbf{R}$ such that: for each x, the function $y \mapsto f(x,y)$ is Riemann integrable with $\int_0^1 f(x,y)\,dy = 0$ (from which it follows that $\int_0^1 (\int_0^1 f(x,y)\,dy)\,dx = 0$); but there exist y such that the function $y \mapsto f(x,y)$ is not Riemann integrable (whence $\int_0^1 (\int_0^1 f(x,y)\,dx)\,dy$ doesn't even make sense).

3.–7. Solve Problems 9 through 13 on pages 290–291. Generalize #13 to the integral of $\prod_{i=1}^k x_i^{r_i}$ over the set of $(x_1, \ldots, x_k)$ with each $x_i \geq 0$ and $\sum_{i=1}^k x_i^{s_i} = 1$. The r_i, s_i can be any real numbers with $r_i > -1$ and $s_i > 0$. (The resulting formula is due to Dirichlet.) In particular, determine the volume of the unit ball in $\mathbf{R}^k$ as a function of k; check that your answer agrees with the known cases $k = 1, 2, 3$. Note what happens to this volume as $k \to \infty$!

8. Let $Q : \mathbf{R}^n \to \mathbf{R}$ be a positive-definite form. Show that the integral of $e^{-Q(x)}$ over $x \in \mathbf{R}^n$ converges, and evaluate this integral. For $y \in \mathbf{R}^n$ determine the integral of $\exp(-Q(x) + i\langle x, y\rangle)$ over $x \in \mathbf{R}^n$, where $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ is the usual inner product.

9. Let n be a positive integer, V the vector space of symmetric $n \times n$ matrices, and $E \subset V$ the open set of positive-definite matrices. Prove that the function $A \mapsto 1/\det(A)$ on E is logarithmically convex.

This problem set due Friday, March 7, at the beginning of class.

[1] *Definitions of Terms Commonly Used in Higher Math*, R. Glover et al. Note that this does **not** define an equivalence relation.