Math 55b: Honors Advanced Calculus and Linear Algebra

Homework Assignment #4 (21 February 2003):
Multivariate differential calculus

> Mathematicians have long regarded it as demeaning to work on problems related to elementary geometry in two or three dimensions, in spite of the fact that it is precisely this sort of mathmetiacs which is of practical value.
> — B. Grünbaum and G.C. Shephard, in *Handbook of Applied Mathematics*

[But then there isn't much new to be done in 2D or 3D geometry, and higher-dimensional geometry can have practical uses too — see the last two problems.]

1. Solve Problems 7 and 10 on page 239.

2. Solve Problems 17 and 18 on page 241 (and see #5 below).

3. [Multivariate Taylor expansions] solve Problem 30 on pages 243–4. (NB What Rudin calls "$\mathcal{C}^{(m)}(E)$" is our "$\mathcal{C}^m(E, \mathbf{R})$"; as usual this works with $\mathbf{R}$ replaced by an arbitrary normed vector space.)

4. The *Laplacian* Δf of a $\mathcal{C}^2$ function f on $E \subset \mathbf{R}^n$ is defined by

$$\Delta f := \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2} = \sum_{j=1}^{n} D_{jj} f.$$

Let $A : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation. Show that $\Delta(f \circ A) = (\Delta f) \circ A$ for all $f \in \mathcal{C}^2$ if and only if A is orthogonal. In particular, the "harmonic functions" (those in the kernel of Δ, i.e. $\mathcal{C}^2$ functions f with $\Delta f = 0$) are preserved by orthogonal changes of variable.

5. [Cauchy-Riemann equtions] The usual identification of $\mathbf{C}$ with $\mathbf{R}^2$ (identify the complex number $z = x + iy$ with the vector (x, y)) lets us regard any map $w : \mathbf{C} \to \mathbf{C}$ as a map of $\mathbf{R}^2$ to $\mathbf{R}^2$, or equivalently as a pair of real-valued functions $u(x, y) = \operatorname{Re} f(x + iy)$, $v(x, y) = \operatorname{Im} f(x + iy)$ on $\mathbf{R}^2$. Prove the following criterion for a $\mathcal{C}^1$ function w to be differentiable *as a map from* $\mathbf{C}$ *to* $\mathbf{C}$, i.e., for there to exist a function $w' : \mathbf{C} \to \mathbf{C}$ such that $w(z+h) = w(z) + hw'(z) + o(|h|)$ as $h \to 0$: the functions u, v must satisfy the *Cauchy-Riemann equations*

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Show that this is the case if w is any polynomial in z with complex coefficients, or the exponential function $w(z) = e^z = e^x(\cos y + i \sin y)$. Prove that such u, v, w are necessarily harmonic functions on $\mathbf{C}$. [It is known that conversely every harmonic function on $\mathbf{C}$ is the real part of a differentiable function of a complex variable, but we shall not go in that direction in Math 55b!]

6. [A foray into algebraic geometry] Recall that as an example of an implicit function we used the "folium of Descartes" consisting of all $(x, y) \in \mathbf{R}^2$ such that $x^3 + y^3 = 3xy$. For $y \neq 0$ let $t = x/y$ and solve for x, y as functions of t. Use this and the Inverse Function Theorem to obtain another proof of the differentiability of y with respect to x at all but two points of the "folium". Explain what happens at those two points in terms of these functions $x(t), y(t)$. [Careful around $(0, 0)$!]

7. Let E be the subspace of the $n+1$-dimensional real vector space $\mathcal{P}_n$ consisting of the polynomials of degree n, i.e. $E = \{\sum_{j=0}^{n} a_j T^j : a_n \neq 0\}$. Fix $P_0 \in E$ and a root t_0 of P_0. Give necessary and sufficient conditions on P_0, t_0 for there to exist a $\mathcal{C}^1$ real-valued function t on a neighborhood of P_0 such that $t(P_0) = t_0$ and $t(P)$ is a root of P for each P in the neighborhood. What is the derivative $t'(P_0)$?

8. Let M_0 be an $n \times n$ matrix, and $\lambda_0 \in \mathbf{R}$ an eigenvalue of M_0 that is a simple root of the characteristic polynomial of M_0. Let v_0 be a λ_0-eigenvector of M_0. Regard M_0 as an element of the n^2-dimensional vector space $\mathcal{M}_n$ of $n \times n$ matrices. Prove that there is a neighborhood U of M_0 in $\mathcal{M}_n$ and $\mathcal{C}^1$ functions $\lambda : U \to \mathbf{R}$ and $v : U \to \mathbf{R}^n$ such that $\lambda(M_0) = \lambda_0$, $v(M_0) = v_0$, and, for all $M \in U$, $\lambda(M)$ is an eigenvalue of M with eigenvector v_0. What is the derivative $\lambda'(M_0)$? What can you say about $v'(M_0)$?

This problem set due Friday, 28 February, at the beginning of class.