Math 55b: Honors Advanced Calculus and Linear Algebra

Homework Assignment #3 (Valentine's Day (Feb.14), 2003):
More univariate calculus, and Stone-Weierstrass

> Fejér discovered his theorem[1] at the age of 19, Weierstrass published [his Polynomial Approximation Theorem] at 70. With time the reader may come to appreciate why many mathematicians regard the second circumstance as even more romantic and heart warming than the first.[2]

More about power series:

1. Our two proofs of formula (5) on p.173 (termwise differentiation of power series inside the circle of convergence) used special properties of calculus over $\mathbf{R}$: the Mean Value Theorem and the Fundamental Theorem of Calculus. Give a direct proof that applies equally well to power series over $\mathbf{C}$ or the field $\mathbf{Q}_p$ of p-adic numbers.

2. For p-adic numbers a_n $(n = 1, 2, 3, \ldots)$, prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $a_n \to 0$ in $\mathbf{Q}_p$. For which $x \in \mathbf{Q}_p$ does the exponential series $E(x) = \sum_{n=1}^{\infty} x^n/n!$ converge? Which $a \in \mathbf{Q}_p$ can be written as $E(x)$ for some $x \in \mathbf{Q}_p$ such that the sum for $E(x)$ converges?

Some integration techniques. First we show how to integrate an arbitrary rational function:

3. [Partial fractions[3]] Let k be an algebraically closed field. Let $K = k(x)$, the field of rational functions in one variable x with coefficients in k. Show that the following elements of K constitute a basis for K as a vector space over k: x^n for $n = 0, 1, 2, 3, \ldots$, and $1/(x-x_0)^n$ for $x_0 \in K$ and $n = 1, 2, 3, \ldots$. (Linear independence is easy. To prove that the span is all of K, consider for any polynomial $Q \in k[x]$ the subspace $V_Q := \{P/Q : P \in k[x], \deg(P) < \deg(Q)\}$ of K, and compare its dimension with the number of basis vectors in V_Q.)

4. Prove that $\tan(x) := \sin(x)/\cos(x)$ is an increasing function on $(-\pi/2, \pi/2)$ mapping this interval bijectively to $\mathbf{R}$. Prove that the inverse map $\tan^{-1}(x)$ has derivative $1/(x^2+1)$. Use this to determine $\int_0^1 (x-x^2)^4 dx/(x^2+1)$. What does this tell you about π?

5. Prove that the integral of any $f \in \mathbf{R}(x)$ is a rational function plus a linear combination of functions of the form $\log|x - x_0|$, $\log((x-x_0)^2 + c)$, and $\tan^{-1}(ax+b)$ $(x_0, a, b, c \in \mathbf{R},\ c > 0)$.

Next we derive some classical product formulas and integrals. Be careful about justifying all steps!

6. Prove that $\int_0^{\pi/2} \cos^n x\, dx = \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x\, dx$ for all $n \geq 2$. Deduce that

$$\int_0^{\pi/2} \cos^n x\, dx = \begin{cases} \frac{2}{3}\frac{4}{5}\frac{6}{7} \cdots \frac{n-1}{n}, & \text{if } n \text{ is odd;} \\ \frac{\pi}{2}\frac{1}{2}\frac{3}{4}\frac{5}{6} \cdots \frac{n-1}{n}, & \text{if } n \text{ is even.} \end{cases}$$

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[1] On Fourier series; see Rudin, pages 199–200 for a sneak preview.

[2] Körner, *Fourier Analysis*, p.294 (conclusion of Chapter 59: "Weierstrass's proof of Weierstrass's theorem").

[3] The decomposition of any $f \in K$ as a linear combination of the basis elements described in this problem is called the "partial fraction decomposition" of f.

7. It follows that

$$\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\cdots\frac{2m}{2m-1}\frac{2m}{2m+1}\cdot\frac{\int_0^{\pi/2}\cos^{2m}x\,dx}{\int_0^{\pi/2}\cos^{2m+1}x\,dx}.$$

Show that

$$1 < \frac{\int_0^{\pi/2}\cos^{2m}x\,dx}{\int_0^{\pi/2}\cos^{2m+1}x\,dx} < \frac{\int_0^{\pi/2}\cos^{2m-1}x\,dx}{\int_0^{\pi/2}\cos^{2m+1}x\,dx} = 1 + \frac{1}{2m},$$

and therefore

$$\frac{\pi}{2} = \lim_{m\to\infty}\left(\frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\cdots\frac{2m}{2m-1}\frac{2m}{2m+1}\right).$$

[This is usually written as the "infinite product"

$$\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\cdots,$$

attributed to Wallis.]

8. Use the formulas of the previous problem to prove that

$$\lim_{n\to\infty}\int_0^{\sqrt{n}\pi/2}\cos^n\frac{x}{\sqrt{n}}\,dx = \sqrt{\pi/2}.$$

Now show that $\lim_{n\to\infty}\cos^n(x/\sqrt{n}) = \exp(-x^2/2)$ for any $x \geq 0$, and use this to prove that[4]

$$\int_0^\infty e^{-x^2/2}\,dx = \sqrt{\pi/2}.$$

Finally, some (Stone-)Weierstrass stuff:

9. i) Suppose $f : [a,b] \to \mathbf{R}$ is a continuous function such that $\int_a^b f(x)x^n dx = 0$ for each $n = 0, 1, 2, 3, \ldots$. Prove that f is the zero function. [This is problem 20 on page 169; it also appeared — without the hint provided there — on a Putnam exam many years ago.]
   ii) Suppose $\alpha, \beta : [0,1] \to \mathbf{R}$ are increasing functions such that there exists n_0 with $\int_0^1 x^n d\alpha(x) = \int_0^1 x^n d\beta(x)$ for each integer $n \geq n_0$. Prove that $\alpha_+ - \beta_+$ and $\alpha_- - \beta_-$ are constant functions on $[0,1)$ and $(0,1]$ respectively, where $\alpha_\pm(x) := \lim_{t\to x\pm}\alpha(t)$ and $\beta_\pm$ is defined in the same way.
   iii) Solve Problem 21 on page 169.

This problem set due Friday, 21 February, at the beginning of class.

[4] As noted in class, it is remarkable that this ubiquitous definite integral can be evaluated in closed form, considering that the indefinite integral $\int \exp(cx^2)\,dx$ cannot be simplified. We shall give another proof of this result when we come to the change of variable formula for multiple integrals.