Math 55b: Honors Advanced Calculus and Linear Algebra

Homework Assignment #11 (21 April 2003):
Fourier III and last

> [This argument] will only carry conviction to those who believe that the infinite sum of small things is always small (except when it is large).[1]

Discrete Fourier analysis provides a good framework for "Gauss sums" and "Jacobi sums" which are ubiquitous in number theory. For our purposes we shall define them as follows. Fix a prime number p, and let ζ be the primitive p-th root of unity $e^{2\pi i/p}$. A *character* is a homomorphism from the multiplicative group $(\mathbf{Z}/p)^* = (\mathbf{Z}/p) - \{0\}$ of order $p-1$ to the $(p-1)$-st roots of unity. The *Gauss sum* associated to a character ψ is

$$G(\psi) := \sum_{n=1}^{p-1} \zeta^n \psi(n).$$

The *Jacobi sum* associated to a pair ψ_1, ψ_2 of characters is defined by

$$J(\psi_1, \psi_2) := \sum_{n=2}^{p-1} \psi_1(n) \psi_2(1-n).$$

For instance, if $p = 5$ and ψ takes ± 1 to 1 and ± 2 to -1 then $G(\psi) = \zeta - \zeta^2 - \zeta^3 + \zeta^4 = \sqrt{5}$ and $J(\psi, \psi) = -1 + 1 - 1 = -1$.

1. Let ψ be a nontrivial character (the "trivial character" sends every element of $(\mathbf{Z}/p)^*$ to 1), and extend it to a complex-valued function on $\mathbf{Z}/p$ by defining $\psi(0) = 0$. The Gauss sum $G(\psi)$ is one value of the discrete Fourier transform of this function. Determine its discrete Fourier transform at all elements of $\mathbf{Z}/p$.

2. Prove that
$$J(\psi_1, \psi_2) = G(\psi_1) G(\psi_2) / G(\psi_1 \psi_2)$$
provided none of ψ_1, ψ_2, $\psi_1\psi_2$ is the "trivial character" sending every element of $(\mathbf{Z}/p)^*$ to 1. What happens if one or more of these characters is trivial? [Hint: remember our formula for $B(x, y)$ and its proof.]

[1] Körner, *Fourier Analysis*, p.296 (in Chapter 60). By now I hope none of our 55b class believes this; keep in mind (also for the take-home final) that I do not believe it either!

3. Prove that $|G(\psi)|^2 = p$ for all nontrivial ψ. If moreover ψ is "real" (sends each element of $(\mathbf{Z}/p)^*$ to either $+1$ or -1), show that $G(\psi) = \pm\sqrt{p}$ if $p \equiv 1 \bmod 4$ and $G(\psi) = \pm i\sqrt{p}$ if $p \equiv -1 \bmod 4$. Numerically compute $G(\psi)$ for nontrivial real characters $\psi \bmod p$ for enough values of p that you detect a pattern in the choices of sign.

I do not ask you to prove this pattern; this sign problem occupied Gauss for years! Our last problem on these sums may suggest one way to solve it:

4. Let N be any positive integer and $\zeta = e^{2\pi i/N}$. Let M be the $N \times N$ matrix whose (a, b) entry is ζ^{ab}; that is, M is the matrix for the discrete Fourier transform mod N. Show that $M^4 = N^2 I_n$. Deduce that each eigenvalue of M is $\pm N^{1/2}$ or $\pm i N^{1/2}$. Conclude that there exist integers r_N, s_N such that $\sum_{a=1}^{N} \zeta^{a^2} = (r_N + i s_N)\sqrt{N}$. Again, compute r_N, s_N for enough small N until you can guess a pattern. How much of this pattern can you prove?

Here's one of many applications of Poisson summation:

5. Fix $c > 0$ and define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = 1/(x^2 + c^2)$. For $t \in \mathbf{R}$ define $F(t) := \sum_{n\in\mathbf{Z}} f(t + 2\pi n)$. Prove that F is a differentiable function of period 2π, and thus can be regarded as a differentiable function on $\mathbf{T}$. Determine its Fourier series (NB we know $\hat{f}$), and deduce the value of $F(0) = \sum_{n\in\mathbf{Z}} f(2\pi n)$. [Your answer should agree with your result for the first problem of the tenth assignment.] Generalize.[2]

The final batch of problems develops a proof of the memorable theorem of Müntz, which answers the following question: Fix an increasing sequence $n_0 < n_1 < n_2 < \cdots$ of nonnegative real numbers and let V be the $\mathbf{R}$-vector space of functions generated by t^{n_i}, $i = 0, 1, 2, \ldots$ For what $\{n_i\}$ is V dense (A) in $L_2([0, 1])$, (B) in $\mathcal{C}[0, 1]$? Müntz's Theorem asserts:

A) V is dense in $L_2([0, 1])$ <u>iff</u> $\sum_{i=1}^{\infty} 1/n_i$ diverges.

B) V is dense in $\mathcal{C}[0, 1]$ <u>iff</u> $n_0 = 0$ and $\sum_{i=1}^{\infty} 1/n_i$ diverges.

Note that the $L_2([0, 1])$ and $\mathcal{C}[0, 1]$ versions of the Weierstrass Approximation Theorem are the special case $n_i = i$ of Müntz; we assume the Weierstrass theorem in the following proof.

6. For any vectors $x_1, x_2, \ldots, x_m$ in a real Hilbert space $\mathcal{H}$, let $\Delta_m(x_1, \ldots, x_m)$ be the determinant of the $m \times m$ matrix whose ijth entry is the inner product of x_i with x_j. Recall that if the x_i are linearly independent then this matrix is positive definite, so in particular $\Delta_m(x_1, \ldots, x_m)$ is positive. In this case let V_m be the

[2] This generalizes in many directions; e.g., if $f : \mathbf{Z}/12\mathbf{Z} \to \mathbf{C}$, what is $f(0) + f(3) + f(6) + f(9)$ in terms of $\hat{f}$?

m-dimensional subspace spanned by the x_i, and show that for any vector $y \in \mathcal{H}$ the distance from y to the nearest point of V_m (that is, the norm of the projection of y to the orthogonal complement $V_m^\perp$) is the square root of the ratio

$$\Delta_{m+1}(x_1, \ldots, x_m, y) \big/ \Delta_m(x_1, \ldots, x_m).$$

[Note that this problem only uses the finite-dimensional space generated by y and the x_i's; the full Hilbert space $\mathcal{H}$ is needed only for what follows.]

7. Taking $\mathcal{H} = L_2([0,1])$, $x_{i+1} = t^{n_i}$ and $y = t^k$ in problem 7 we find determinants Δ_m, Δ_{m+1} of the form $\det(1/(a_i + b_j))_{i,j=1}^M$. Prove that, for any real numbers $a_1, \ldots, a_M; b_1, \ldots, b_M$ such that none of the $a_i + b_j$ vanishes, the value of this determinant is

$$D_M(a_1, \ldots, a_M) D_M(b_1, \ldots, b_M) \Big/ \prod_{i=1}^{M} \prod_{j=1}^{M} (a_i + b_j)$$

where $D_M(r_1, \ldots, r_M) = \prod_{1 \le i < j \le M} (r_i - r_j)$. Use this to compute the L_2 distance from x^k to the space V_m spanned by x^{n_i}, $0 \le i < m$. (Why are these m vectors linearly independent?)

8. Conclude that, provided k is not one of the n_i, the $L_2([0,1])$ closure of $V = \cup_{m=1}^{\infty} V_m$ contains x^k if and only if $\sum_{i=1}^{\infty} 1/n_i$ diverges. Use this to deduce part A of Müntz's Theorem.

9. The "only if" half of part B is now easily accessible: prove that if $n_0 > 0$ or $\sum_{i=1}^{\infty} 1/n_i < \infty$ then $\mathcal{C}[0,1]$ contains functions not in the closure of V. To get the reverse implication we need one more trick: for any $f \in L_2([0,1])$ define $\int f : [0,1] \to \mathbf{R}$ by $\int f(x) = \int_0^x f(t)\,dt$, i.e., the inner product of f with the characteristic function of $[0,x]$. As part of last problem of PS8, we showed in effect that $\int$ is a continuous linear map of norm ≤ 1 from $L_2([0,1])$ to $\mathcal{C}[0,1]$. Use this map to finish the proof of Müntz's Theorem.

This problem set is due Friday, May 2 in class.