Math 55b: Honors Advanced Calculus and Linear Algebra

Homework Assignment #1 (29 January 2003):
Univariate differential calculus

> It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out. — E. Artin, *Geometric Algebra.*

First, a few problems on "differential algebra"; that is, familiar algebraic axioms of a (usually commutative) ring or field, extended by a map $D : f \mapsto f'$ satisfying the axioms $(f+g)' = f' + g'$ and $(fg)' = fg' + f'g$. Such a map is called a *derivation* of the ring or field. Note that in the case of a field, the formula $(f/g)' = (f'g - fg')/g^2$ holds automatically because the argument we gave in class starting from $f = g(f/g)$ uses only the field and derivation axioms. The topological considerations that arise in the definition of the derivative enter into some of the following problems but are not the main point.

1. i) If $f, g, h : [a, b] \to \mathbf{R}$ are differentiable at $x \in [a, b]$, prove that so is their product fgh, and find $(fgh)'(x)$.
   ii) If $f, g : [a, b] \to \mathbf{R}$ are thrice differentiable at $x \in [a, b]$, prove that so is their product fg, and find $(fg)'''(x)$.
   iii) Generalize.

2. Let V be a finite dimensional vector space over $\mathbf{R}$ or $\mathbf{C}$. Consider a function $f : [a, b] \to \mathcal{L}(V)$, and let $x \in [a, b]$. Assume that $f(x)$ is invertible and f is differentiable at x. Thus $g(x) := (f(x))^{-1}$ exists in a neighborhood of x (possibly one-sided, if $x = a$ or $x = b$), and is continuous at x. Assume that g is differentiable at x — we'll prove this later when we develop differential calculus in several variables. Determine $g'(x)$. [Hint: extend Thm. 5.3, and remember Artin's quote above.]

3. [Wronskians[1]] The numerator $f'g - fg'$ of the formula for f/g is the case $n = 2$ of a *Wronskian.* In general, if $f_1, \ldots, f_n$ are scalar-valued functions on $[a, b]$ differentiable $n-1$ times at some $x \in [a, b]$, their "Wronskian" at x is the determinant of the $n \times n$ matrix whose (i, j) entry is the $(j-1)$-st derivative of f_i. Suppose each f_i is differentiable $n-1$ times on all of $[a, b]$. Prove that if the f_i are linearly dependent over the scalar field then their Wronskian vanishes. Is the converse true? What if the f_i are polynomials?

4. i) Prove that if K is a field equipped with a derivation D then $k := \ker D$ is a subfield of K. This is called the "constant subfield" of K. Show that $D : K \to K$ is k-linear.
   ii) Now suppose $K = F(X)$, the field of rational functions in one variable over some field F. Define $D : K \to K$ by the usual formula: if $P = \sum_n a_n X^n$ then $D(P) = \sum_n n a_n X^{n-1}$; and any $f \in K$ is the quotient P/Q of two polynomials,

[1] I believe that this is pronounced as if it were "Vronskians", but I could be vrong.

so we may write $D(P/Q) = (P'Q - PQ')/Q^2$. Show that this is well-defined (i.e. if $f = P_1/Q_1 = P_2/Q_2$ then the two definitions of $D(f)$ agree), and yields a derivation of K. What is the constant subfield?

Next, a few problems on differential analysis. There are lots more neat problems in the textbook, most of which do not depend on omitted material such as L'Hôpital's rule and Thm. 5.12.[2]

5. [Lipshitz condition; cf. problem 1 on p.114]
   i) Suppose $f : [a,b] \to \mathbf{R}$ is differentiable on $[a,b]$ and $|f'|$ is bounded on $[a,b]$. Prove that $f(x) - f(y) = O(|x-y|)$ for all $x, y \in [a,b]$. (Recall that $f = O(g)$ means that there exists $M \in \mathbf{R}$ such that $|f| \leq Mg$; an equivalent notation is $f \ll g$.) Show that not all functions $f : [a,b] \to \mathbf{R}$ satisfying $f(x) - f(y) = O(|x-y|)$ are differentiable.
   ii) In general, if for some $p \geq 0$ we have $f(x) - f(y) = O(|x-y|^p)$ then f is said to satisfy the *Lipshitz condition* with exponent p. Let Λ_p be the set of all such f, which is clearly a vector space. For instance, Λ_0 consists of all bounded functions; if $p > 0$ then all functions in Λ_p are continuous; if $p > q$ then $\Lambda_p \subseteq \Lambda_q$; and in part (i) we showed that all differentiable functions are in Λ_1 but not conversely. If $0 \leq q < p \leq 1$, show that Λ_q strictly contains Λ_p. If $p > 1$, describe Λ_p.

6. Solve problems 2,3 on page 114. [#2 is essentially the inverse function theorem in dimension 1.]

7. Solve problem 27 on page 119 (which also requires problem 26). Note that we do not yet prove existence of f.

8. Assume the existence of a differentiable function $\exp : \mathbf{R} \to \mathbf{R}$ such that $\exp'(x) = \exp(x)$ for all x and $\exp(0) = 1$. Show that $\exp(x+y) = \exp(x)\exp(y)$ for all $x, y \in \mathbf{R}$. Deduce that $\exp(x) > 0$ for all $x \in \mathbf{R}$, and $x^n = o(\exp(x))$ as $x \to +\infty$ for each $n = 1, 2, 3, \ldots$. (L'Hôpital is not needed!)

9. [A nonzero function with zero Taylor series.] Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = 0$ for $x \geq 0$ and $f(x) = \exp(1/x)$ for $x < 0$. Prove that f is infinitely differentiable and that all its derivatives vanish at $x = 0$. Can you obtain an explicit formula for the n-th derivative?

This problem set due Friday, 7 February, at the beginning of class.

[2] Apropos Thm. 5.12, "discontinuity of the first/second kind" is defined on page 94; this is not standard terminology.