Math 55b: Honors Advanced Calculus and Linear Algebra

Introduction to Hilbert Space
II: Separable inner-product spaces;
orthogonal complements, projections, and duality

The space l_2 is much larger than any of the finite-dimensional Hilbert spaces $\mathbf{F}^n$ — for instance, it is not locally compact — but it is still small enough to be "separable"; this in fact topologically characterizes l_2. This notion is defined as follows:

Definition. *A metric space is* separable *if it contains a dense countable subset.*

(We consider a finite or even empty set to be countable. The phrase "second countable" is sometimes used for "separable".)

Examples. Any compact set is separable: for each $n = 1, 2, 3, \ldots$ there is a finite $(1/n)$-net, and the union of these nets over n is a countable dense set. The metric space $\mathbf{R}$ is separable because $\mathbf{Q} \subset \mathbf{R}$ is a dense countable subset. The direct sum of two separable spaces is separable, because the Cartesian product of two countable sets is countable. Thus in particular $\mathbf{C}$, so also $\mathbf{R}^n$ and $\mathbf{C}^n$, are separable. Even l_2 is separable, because $l_2^{(0)}$ is dense in l_2 and $l_2^{(0)}$ is a countable union of separable subsets $\mathbf{F}^n$.

If X is a separable metric space, and $S \subset X$ is a subset such that there exists $d_0 > 0$ with $d(s, s') \geq d_0$ for all distinct $s, s' \in S$, then S is countable. Indeed, for each $\epsilon > 0$, we can write X as the countable union of ϵ-neighborhoods (specifically, of ϵ-neighborhoods centered at the points of its dense countable subset), and once $\epsilon < d/2$ each of these neighborhoods contains at most one point of S. Thus conversely if X is a metric space containing an *un*countable subset S any two of whose points are at least d_0 apart, then X is not separable. For example, a discrete space is separable if and only if it is countable. The normed space l_∞ of bounded sequences $(a_1, a_2, \ldots)$ of scalars with the sup norm is not separable, because it contains the uncountable subset $S = \{\{a_n\}_{n=1}^\infty :$ each $a_n = 0$ or $1\}$ with $d_0 = 1$.

Theorem. *Let V be an inner product space. Then V is separable if and only if it has a countable ontb. In this case every orthogonal set in V is countable.*

Proof: If V is separable, let $\{v_n\}_{n=1}^\infty$ be a dense sequence in V. Clearly $\{v_n\}$ spans V topologically. Discard each v_n that is a linear combination of $v_1, \ldots, v_{n-1}$; the resulting linearly independent sequence may no longer be dense, but its linear span is the same, so is still dense in V. Now apply Gram-Schmidt to obtain an orthonormal sequence w_n still with the same linear span, so the w_n constitute a countable ontb.

Conversely, if V has a countable ontb it is either isometric with $\mathbf{F}^n$ or has a dense subset isometric with $l_2^{(0)}$, and we have shown that each of these spaces is separable.

Finally, if S is any orthonormal subset of an inner product space then any two of its points are at the same distance, namely $\sqrt{2}$. Thus if the space is separable then S must be countable. By normalization the same is true also of orthogonal sets. $\square$

Corollary. *Every separable Hilbert space is isometric with either* $\mathbf{F}^n$ *(some* $n = 0, 1, 2, \ldots$*) or* l_2*.*

Orthogonal projections and complements in Hilbert space. Most uses of the completeness of a Hilbert space go through the following results, which show that orthogonal projections and complements work for a Hilbert space as they do for a finite-dimensional inner product space. In particular, we can identify a Hilbert space $\mathcal{H}$ with its topological dual $\mathcal{B}(\mathcal{H}, \mathbf{F})$ as we did for a finite-dimensional inner-product space.

Theorem. *Let* V *be an inner product space, and* $W \subset V$ *a complete subspace. Then for each* $v \in V$ *there exists a unique* $P(v) \in W$ *such that* $|v - P(v)| = \min_{w \in W} |v - w|$*.*

Proof: Since $\{|v - w| : w \in W\}$ is a nonempty set bounded below, it has an infimum, call it d. Let $\{w_n\}$ be any sequence in W such that $|v - w_n| \to d$. We shall show that w_n is necessarily a Cauchy sequence. Since W is assumed complete, it will follow that $\{w_n\}$ converges to some $w \in W$. Then $|v - w| = d$, so we may set $P(v) = w$.

Suppose that $u_1, u_2 \in W$ with $|v - u_i|^2 \leq d^2 + \epsilon^2$. By the parallelogram law,

$$\left| v - \frac{u_1 + u_2}{2} \right|^2 = \frac{1}{2}\big(|v - u_1|^2 + |v - u_2|^2\big) - \frac{1}{4}|u_1 - u_2|^2$$

But the left-hand side is at least d^2, and the right hand side at most $d^2 + \epsilon^2 - \frac{1}{4}|u_1 - u_2|^2$. Thus $|u_1 - u_2| \leq 2\epsilon$. In particular, taking $\epsilon = 0$, we see that if $P(v)$ exists then it is unique.

Now for each $\epsilon > 0$ there exists N such that $|v - w_n|^2 \leq d^2 + \epsilon^2$ for all $n > N$. Thus $|w_m - w_n| \leq 2\epsilon$ for all $m, n > N$, and we're done. $\square$

As in the finite-dimensional case, it is then readily seen that $v - P(v) \subset W^\perp$. We thus express any $v \in V$ as a sum of vectors in W and $W^\perp$. This representation is unique because $W \cap W^\perp = \{0\}$. It follows that $P : V \to W$ is a linear transformation. This transformation is called *(orthogonal) projection* to W.

Corollary. *Under these hypotheses,* V *is the orthogonal direct sum of* W *and* $W^\perp$*, and* $W = (W^\perp)^\perp$*.*

Note that for any $W \subset V$, the orthogonal complement $W^\perp$ is a closed subspace of V, and is the same as $\overline{W}^\perp$ where $\overline{W}$ is the closure of W in V. Thus $(W^\perp)^\perp$ is a closed subspace of W containing W. By the above Corollary, if $\overline{W}$ is complete then $(W^\perp)^\perp = \overline{W}$. Without this hypothesis on $\overline{W}$, it could happen that $(W^\perp)^\perp$ is strictly larger than $\overline{W}$. For instance, let $V = l_2^{(0)}$ and let W be the intersection of $l_2^{(0)}$ with the orthogonal complement of $(1, 1/2, 1/3, 1/4, \ldots)$ in l_2.

Now let V be a Hilbert space. Then $W \subset V$ is complete if and only if it is closed. Thus in a Hilbert space $(W^\perp)^\perp = W$ if and only if $W = \overline{W}$.

We can now conclude that in a separable Hilbert space any orthogonal (orthonormal) set is contained in an orthogonal topological basis (ontb). Using the Axiom of Choice the same can be proved for an arbitrary Hilbert space $\mathcal{H}$. We can then define the *dimension* of $\mathcal{H}$ as the cardinality of a basis of $\mathcal{H}$ — once we show that all bases have the same cardinality. We have done this already for a Hilbert space with a countable basis; given the existence of ontb's, it can be done in general without much further difficulty using the fact that $c \cdot \aleph_0 = c$ for any infinite cardinal c.

The *topological dual* of a normed vector space V is the vector space $\mathcal{B}(V, \mathbf{F})$ of *continuous* linear functionals on V. If V is an inner product space then any $v \in V$ may be regarded as the functional $w \mapsto (v, w)$; this is continuous of norm $|v|$ by Cauchy-Schwarz, so we get an isometric embedding of V into V^*. In general the image of V is not all of V^*, since V^* is necessarily complete (why?). However, this is the only obstruction: *a Hilbert space is its own topological dual.* More precisely:

Theorem. *Let V be a Hilbert space and $v^* \in V^*$ a continuous linear functional. Then there exists a unique $v \in V$ such that $v^*(w) = (w, v)$ for all $w \in V$.*

Proof: Uniqueness is easy: if v_1, v_2 are two such v's then $v^*(v_1 - v_2) = 0$ yields $|v_1 - v_2|^2 = 0$ and thus $v_1 = v_2$. We next prove existence. If $v^* = 0$ then we of course take $v = 0$. Else let $W = \ker v^*$. This is a proper closed subspace of V and thus has nonzero orthogonal complement $W^\perp$. We claim that $W^\perp$ is one-dimensional. Else it would contain two linearly independent vectors v_1, v_2; but we can find scalars a_1, a_2 not both zero such that $a_1 v^*(v_1) + a_2 v^*(v_2) = 0$, and then $a_1 v_1 + a_2 v_2$ is a nonzero vector in $W \cap W^\perp$, which is impossible. For the same reason, if v_0 is a nonzero vector in $W^\perp$ then $v^*(v_0) \neq 0$. Choose such v_0, and define

$$v = \frac{v^*(v_0)}{|v_0|^2} v_0.$$

Then $v^*(v_0) = (v, v_0)$. Moreover $v^*(w) = (w, v)$ for all $w \in W$ since both sides vanish. But we showed that v_0 spans $W^\perp$, and thus v_0 together with W span V. Thus $v^*(w) = (w, v)$ for all $w \in V$ and we are done. $\square$