Math 55a: Honors Advanced Calculus and Linear Algebra

Practice Problems (7 January Y2K)

1. Let $\mathbf{R}_d$ be the metric space obtained by using the discrete metric on $\mathbf{R}$, and let $\mathbf{R}_{(3)}$ be $\mathbf{R}$ with the metric
$$d(x,y) = |x^3 - y^3|$$
(convince yourself that this is in fact a legitimate metric). The identity function $x \mapsto x$ then gives rise to six functions between pairs of the three metric spaces $\mathbf{R}$, $\mathbf{R}_d$, $\mathbf{R}_{(3)}$.
i) Which of these six functions $\mathbf{R} \rightleftarrows \mathbf{R}_d$, $\mathbf{R}_d \rightleftarrows \mathbf{R}_{(3)}$, $\mathbf{R}_{(3)} \rightleftarrows \mathbf{R}$ are continuous?
ii) Of those, which are uniformly continuous?

2. [Contraction mapping theorem; cf. the last problem of the Topology IV set.] A function f from a metric space X to itself is said to be a <u>contraction</u> if there exists a constant $c < 1$ such that $d(f(x), f(y)) \leq c d(x,y)$ for all $x, y \in X$ [i.e. f shrinks all distances by a factor of at least $1 : c$]. Prove that f is continuous, and that it has at most one fixed point, i.e. there is at most one $z \in X$ such that $f(z) = z$. Give an example of a nonempty X and a contraction map on X without a fixed point. Show that if X is nonempty and complete then every contraction map has a (necessarily unique) fixed point.

This is a surprisingly powerful result; for instance, next term we shall prove existence and uniqueness of solutions to certain differential equations by applying the contraction mapping theorem on a function space.

3. As in Axler, let $F = \mathbf{R}$ or $\mathbf{C}$, and let V be an n-dimensional space over F. In class we used the determinantal characterization of invertibility to prove that the singular linear transformations of V constitutes a closed subset of $\mathcal{L}(V)$. We then noted a determinant-free alternative proof: fix an inner product on V, and let $T_i \in \mathcal{L}(V)$ converge to T, and assume that each T_i is singular. Choose nonzero $v_i \in \ker T_i$, which may be assumed to have unit norm (why?). The unit ball in V is compact, so a subsequence of $\{v_i\}$ converges to some unit vector v. Then $Tv = 0$, so T is singular, as claimed.
Generalize this to a determinant-free proof that for each $r < n$ the linear transformations of rank at most r constitute a closed subset of $\mathcal{L}(V)$.

4. [Markov processes with all transition probabilities positive.] Let A be an $n \times n$ matrix all of whose entries are positive real numbers and each of whose columns sums to 1. We shall show: 1 is an eigenvalue of A of algebraic multiplicity 1, with eigenspace generated by a vector z with each $z_i < 0$ and $\sum_{i=1}^{n} z_i = 1$; all other complex eigenvalues of A have absolute value < 1. See the comments below for the contexts and a typical application of this result.
i) For $x \in \mathbf{C}^n$ define $|x| = \sum_{i=1}^{n} |x_i|$; recall that this is a norm on $\mathbf{C}^n$. Prove that $|Ax| \leq |x|$ for all $x \in \mathbf{C}^n$, with equality if and only if x is proportional to a vector all of whose coordinates are nonnegative real numbers.
ii) Let D be the set of all vectors x with each x_i a nonnegative real number such that $\sum_{i=1}^{n} x_i = 1$. Show that $Ax \in D$ for all $x \in D$. Thus A restricts to a function from D to D; Show that A has a unique fixed point in D. [This can be done either using Problem 2 above, or the final problem of the Topology IV set.] Let z be that fixed point. Necessarily $z_i > 0$ for each i (why?).
iii) Prove that 1 us an eigenvalue of A of algebraic multiplicity 1 [warning: it is not enough to show that z generates $\ker(I - A)$!], and that $|\lambda| < 1$ for any complex eigenvalue $\lambda \neq 1$ of A.

Together with the Jordan canonical form [Axler, pages 183 ff.], this implies that for any vector $x \in \mathbf{R}^n$ with each $x_i \geq 0$ we have $A^m x \to az$ as $m \to \infty$, where $a = \sum_{i=1}^n x_i$; moreover, convergence is exponential: there exist $c < 1$ and K such that $|A^m x - az| < Kc^m$.

iv) How much of these results remains true if the entries of A are allowed to be either positive or zero?

A typical application of this is as follows. A *Markov chain* is a system that, at time $m = 0, 1, 2, \ldots$, is in one of n states, and its state at time $m+1$ depends only on the state at time m according to a table of "transition probabilities". Let A be the matrix of these transition probabilities; its entries are nonnegative, and each column sums to 1. If v is the vector of state probabilities at time m, then Av is the vector at time $m+1$; thus if x gives the probabilities at time 0 then $A^m x$ does so for time m. If all the transition probabilities are positive, the results of (i)-(iii) apply; if some probabilities may vanish, see (iv).

5. [Hadamard's inequality.] Let A be a real or complex $n \times n$ matrix with column vectors $v_1, \ldots, v_n$. If the v_j's are orthogonal, show that

$$|\det A| = |v_1||v_2| \cdots |v_n|.$$

Prove that for all $n \times n$ matrices A

$$|\det A| \leq |v_1||v_2| \cdots |v_n|,$$

with equality if and only if the v_j's are orthogonal. [Here $|\cdot|$ is the usual inner-product norm, $|v| = \langle v, v\rangle^{1/2}$. Hint: apply Gram-Schmidt to $v_1, \ldots, v_n$.]

In particular, if each entry of A has absolute value ≤ 1 then $|\det A| \leq n^{n/2}$ — note that for $n > 2$ this quickly gets much smaller than the obvious upper bound of $n!$. If equality occurs, A is said to be a "Hadamard matrix". An example is the matrix whose (j,k) entry is $\exp(2\pi ijk/n)$. A *real* Hadamard matrix must have all its entries ± 1; in this case it is easy to see that n must be 1, 2, or a multiple of 4. It is a long-standing conjecture that this necessary condition on the size of a real Hadamard matrix is also sufficient.

6. [Positive-definite matrices; cf. Axler, p.144 ff.] A real symmetric or complex Hermitian $n \times n$ matrix A is said to be *positive definite* if $\langle x, Ax\rangle > 0$ for all vectors x, with equality if and only if $x = \mathbf{0}$. Prove that a square matrix A over $F = \mathbf{R}$ or $\mathbf{C}$ is positive definite if and only if there exists a basis $\{v_i\}$ of F^n such that $A_{ij} = \langle v_i, v_j\rangle$ for each i, j. [Hint: in one direction it will help to observe that $\boldsymbol{\langle} x, y \boldsymbol{\rangle} := \langle x, Ay\rangle$ defines an inner product on F^n.] Note that this is equivalent to the existence of a matrix S such that $A = S^{\mathrm{T}}S$ or $S^{\mathrm{H}}S$. Using the previous problem, conclude that if A is positive definite then $0 < \det A < \prod_{i=1}^n A_{ii}$ with equality if and only if A is diagonal.

7. Let V be a finite-dimensional space over a field F. Linear transformations $A, B \in \mathcal{L}(V)$ are said to be *simultaneously diagonalizable* if there exists a basis for V for which A, B are both diagonal. Show that if this is the case then $AB = BA$. Give an example of V, A, B where A, B commute and are diagonalizable separately but not simultaneously. If either A or B has distinct eigenvalues, show that $AB = BA$ if and only if A, B are simultaneously diagonalizable. Do the same when $F = \mathbf{R}$ or $\mathbf{C}$ and A, B are normal — in which case the basis can be orthonormal. Generalize to three or more pairwise commuting transformations. Can you generalize Schur's theorem [Axler, p.111, #6.28] to commuting linear operators?