Math 55a: Honors Abstract Algebra

Homework Assignment #8 (5^2 October 2017):
Linear Algebra VIII: The spectral theorem; spectral graph theory; symplectic structures

As soon as I get into [Math 55a] class, I'm fighting off a swarm
Of positive-definite non-degenerate symmetric bilinear forms!
—from a somewhat redundantly titled patter-song in *Les Phys* (P. Dong, 2001)

More about self-adjoint operators on $\mathbf{R}^n$ and their spectra:

1. [The right way to prove the Spectral Theorem for self-adjoint operators on $\mathbf{F}^n$.] Let V be an inner-product space over either $\mathbf{R}$ or $\mathbf{C}$, and $T : V \to V$ an operator that is self-adjoint (over $\mathbf{R}$) or Hermitian (over $\mathbf{C}$). We noted in class that if V is the direct sum $\oplus_{j=1}^{m} V_{\lambda_j}$ of eigenspaces for some $\lambda_j \in \mathbf{R}$ then the largest of the λ_j is also $\max\langle Tv, v\rangle$, where the maximum is taken over all $v \in V$ with $\|v\| = 1$. Suppose conversely that some $v_1 \in V$ satisfies $\|v_1\| = 1$ and $\langle Tv_1, v_1\rangle \geq \langle Tv, v\rangle$ for all $v \in V$ with $\|v\| = 1$. Prove that v_1 is an eigenvector of T (necessarily with eigenvalue $\langle Tv_1, v_1\rangle$). [Hint: One approach is to construct a semidefinite pairing on V and use problem 3 on the previous problem set.]

Once we develop the topological notion of compactness, we'll be able to deduce from this (and without invoking the Fundamental Theorem of Algebra!) the existence of an eigenvector for every self-adjoint operator on a finite-dimensional inner product space, and thus to prove the spectral theorem for such operators.

2. Let $A = (a_{ij})$ be a symmetric real $n \times n$ matrix with nonnegative entries, and λ its largest eigenvalue.
   i) Prove that there is a λ-eigenvector of A all of whose entries are nonnegative, and that $\mu \geq -\lambda$ for all eigenvalues μ of A. (In particular $\lambda \geq 0$.)
   ii) Say A is "disconnected" if there is a partition $\{1, 2, \ldots, n\} = I \cup J$, with I, J (disjoint and) nonempty, such that $a_{ij} = 0$ for all $i \in I$ and $j \in J$. Otherwise we say A is "connected". Prove that if A is connected then the λ-eigenspace has dimension 1 and a generator all of whose entries are strictly positive.
   iii) Suppose A is connected. Give a necessary and sufficient condition for $-\lambda$ to be an eigenvalue of A, and under this condition determine the $(-\lambda)$-eigenspace.

The notions of "(dis)connected" agree with "(dis)connected graph" when A is an adjacency matrix of a graph [see below]. The hypothesis that A be symmetric can be dropped as long as we require $a_{ij} = a_{ji} = 0$ for a disconnection. Then $|\mu| \leq \lambda$ for all complex eigenvalues μ, and if A is connected and $\mu \neq \pm\lambda$ then $|\mu| < \lambda$. I do not assign this extension now because it will fit much more naturally into Math 55b.

Some more spectral graph theory. We recall the basic definitions:

A "graph" $G = (V, E)$ is a finite set V (the "vertices" of G) together with a collection E of two-element subsets of V (the "edges" of G); we say vertices v, v' are "adjacent" if $\{v, v'\} \in E$. The *adjacency matrix* A_G of G is defined as follows: if $V = \{v_1, v_2, \ldots, v_n\}$ then A_G has (j, k) entry 1 if $\{v_j, v_k\} \in E$, and 0 otherwise. This definition depends on the ordering of the vertices, but the underlying linear transformation $T_G : \mathbf{R}^V \to \mathbf{R}^V$ is canonical: T_G takes the unit vector corresponding to each vertex v to the sum of the unit vectors corresponding to vertices v' adjacent to v; in other words, for any $f : V \to \mathbf{R}$ we have $Tf(v) = \sum_{\{v,v'\} \in E} f(v')$. Clearly A_G is symmetric, so T_G is self-adjoint with respect to the inner product $\langle f, g\rangle = \sum_{v \in V} f(v)g(v)$ (and this is also easy to check directly).

3. We noted in class that A_G has trace zero. Prove that the trace of A_G^2 is $2|E|$. Give formulas for the traces of A_G^3 and A_G^4 in terms of the graph structure. Check that for the Petersen graph these formulas are consistent with our determination of the spectrum of A_G (the eigenvalues are 3, 1, and -2, with the corresponding subspaces having dimensions 1, 5, and 4 respectively).

4. The *product* of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \times G_2$ with vertex set $V_1 \times V_2$ (as you expect), and (as you might not expect) vertices (v_1, v_2) adjacent to (v_1', v_2') <u>iff</u> either

- v_1, v_1' are adjacent in G_1 and $v_2 = v_2'$, or
- v_2, v_2' are adjacent in G_2 and $v_1 = v_1'$.

Express the operator $T_{G_1 \times G_2}$ in terms of T_{G_1}, T_{G_2}, and our usual tensor operators. Then check that the resulting description of the spectrum of $T_{G_1 \times G_2}$ in terms of the spectra of T_{G_1} and T_{G_2} is consistent with the formulas in the previous problem for the trace of T_G, T_G^2, and T_G^3.

5. The *hypercube graph* $\mathcal{H}_n$ may be defined for $n = 0, 1, 2, \ldots$ as follows: fix a set I of size n; then the vertices of $\mathcal{H}_n$ are subsets $J \subseteq I$, and two subsets J, J' are adjacent <u>iff</u> one of them is the disjoint union of the other together with a single-element set.
   i) Show that the product of two hypercube graphs $\mathcal{H}_m$ and $\mathcal{H}_{m'}$ is isomorphic with $\mathcal{H}_{m+m'}$, and use this to find the spectrum of $\mathcal{H}_n$ (that is, all eigenvalues of $T_{\mathcal{H}_n}$ and the dimension of each eigenspace). Make sure your answer is consistent with the results of Problem 2.
   ii) Show that if $n > 0$ and S is any set consisting of 2^{n-1} vertices of $\mathcal{H}_n$ (i.e. $S \cup \overline{S}$ is a partition of the vertex set into two parts of equal size) then there are at least 2^{n-1} edges $\{v, v'\}$ with $v \in S$ and $v' \notin S$. Find all S for which equality holds, and prove that there are no others. (Hint: apply the result quoted in Problem 1, but not quite to the 2^n-dimensional space on which T acts.)

The fact that $\mathcal{H}_m \times \mathcal{H}_{m'} \cong \mathcal{H}_{m+m'}$ might also make this notion of "graph product" feel more natural.

The last three problems concern *symplectic spaces*, which for our purposes will be finite-dimensional vector spaces equipped with a nondegenerate, bilinear, alternating pairing. Recall that "alternating" means $\langle v, v \rangle = 0$ for all vectors v, which by the usual "polarization" trick implies $\langle w, v \rangle = -\langle v, w \rangle$ for all vectors v, w. Such pairings arise often in higher mathematics; they might not be as intuitive as inner products and other symmetric pairings, but fortunately their structure is even simpler than that of symmetric pairings. We begin by showing one way that nondegenerate alternating pairings arise naturally; you've seen another in Problem Set 4.

6. Let $\langle \cdot, \cdot \rangle$ be a sesquilinear Hermitian pairing on a complex vector space V. Prove that the imaginary part of $\langle \cdot, \cdot \rangle$ is an $\mathbf{R}$-bilinear alternating pairing on V, and is nondegenerate if and only if $\langle \cdot, \cdot \rangle$ is.

Likewise for the real part, which is symmetric rather than alternating, but that's more or less what we'd expect by now; I don't ask you to prove these analogous results for the real part because the proofs for the real and imaginary parts are so similar.

If you remember "polarities" from Problem Set 4, you know that it is no accident that this example produces only symplectic spaces of even dimension. In the next problem, you'll prove this for all symplectic spaces, and moreover show that all symplectic structures on F^{2n} are equivalent. This is similar enough to what we've done with inner-product spaces that I refrain from the usual practice of pre-chewing the proof into bite-sized pieces.

7. i) Let V be a finite-dimensional vector space over a field F, and $\langle \cdot, \cdot \rangle$ be a nondegenerate alternating pairing on V. Show that V has even dimension $2n$ and a basis $(v_1, \ldots, v_n, w_1, \ldots, w_n)$ such that $\langle v_i, w_i \rangle = -\langle w_i, v_i \rangle = 1$ for each i and the pairing of any two basis vectors not of the form (v_i, w_i) or (w_i, v_i) vanishes. [Equivalently, if $v = \sum_i (a_i v_i + b_i w_i)$ and $v' = \sum_i (a_i' v_i + b_i' w_i)$ then $\langle v, v' \rangle = \sum_i (a_i b_i' - a_i' b_i)$.]
   ii) Deduce that if $F = \mathbf{R}$ then V is isomorphic with the symplectic space constructed in Problem 6.
   iii) Obtain a theorem about antisymmetric matrices over F equivalent to (i) and (ii): a $2n \times 2n$ matrix over F is invertible and antisymmetric if and only if it is of the form [fill in the blank] for some $A \in \mathrm{GL}_{2n}(F)$. ("Antisymmetric" includes the condition that all diagonal entries vanish, even in characteristic 2 where this condition is not implied by $M^{\mathrm{T}} = -M$.)

8. Suppose now that F is a finite field of q elements. How many choices are there for the symplectic basis $(v_1, \ldots, v_n, w_1, \ldots, w_n)$? [This is also the size of the symplectic group $\mathrm{Sp}(V, \langle \cdot, \cdot \rangle) \cong \mathrm{Sp}_{2n}(F)$.]

This problem set is due Wednesday, 2^5 October (well, 57^0 November), at the beginning of class.