Math 55a: Honors Abstract Algebra

Homework Assignment #6 (11 October 2017):
Linear Algebra VI: tensors, more eigenstuff, and a bit on inner products

> It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out. — E. Artin, *Geometric Algebra*.

> It is a mathematical fact that the casting of this pebble from my hand alters the centre of gravity of the universe.
> — Thomas Carlyle (1795–1881), who apparently knew his John Donne ("if a clod be washed away by the sea, Europe is the less"[1]) but was not as clear on Newton's Laws. . .

We begin with some basic problems on tensors and tensor products. For the first of these, recall that the "rank" of a linear transformation $T : U \to V$ is the dimension of its image $T(U)$; the *rank* of a matrix is the rank of the linear transformation it represents.

1. Let $\{u_i\}_{i=1}^m$ and $\{v_j\}_{j=1}^n$ be bases of the F-vector spaces U and V, and consider the general element $w = \sum_i \sum_j w_{ij}(u_i \otimes v_j)$ of $U \otimes V$. Prove that w is the sum of r pure tensors if and only if the matrix (w_{ij}) has rank at most r.

2. Let V be a vector space of finite dimension n over a field F. We constructed a linear map, the trace, from $\mathrm{End}(V)$ to F. Hence the map from $\mathrm{End}(V) \times \mathrm{End}(V)$ to F taking (S, T) to the trace of ST is bilinear. Prove that it is symmetric. For what n can there exist $S, T \in \mathrm{End}(V)$ such that $ST - TS$ is the identity map? (By comparison, recall that the operators $P \mapsto dP/dz$ and $P \mapsto zP$ on the infinite-dimensional space $\mathcal{P} = F[z]$ satisfy $ST - TS = I$.)

Tensors and eigenstuff:

3. Fix $a \in \mathbf{C}$, and let $T : \mathbf{C} \to \mathbf{C}$ be the map $z \mapsto az$. This is an $\mathbf{R}$-linear operator, so we may consider the linear operator $T' = T \otimes 1$ on the complex vector space $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$. What are the eigenvalues and eigenvectors of T'? (Warning: the answer depends on whether $a \in \mathbf{R}$.)

4. Let U, V be vector spaces over a field F, equipped with linear operators $S \in \mathrm{End}(U)$, $T \in \mathrm{End}(V)$. Consider $S \otimes T \in \mathrm{Hom}(U \otimes V)$.
   i) If $\lambda \in F$ is an eigenvalue of S, and $\mu \in F$ is an eigenvalue of T, prove that $\lambda\mu$ is an eigenvalue of $S \otimes T$.
   ii) If U, V are finite dimensional and F is algebraically closed, prove that every eigenvalue of $S \otimes T$ it the product of an eigenvalue of S with an eigenvalue of T.

   iii) Show, by constructing a counterexample with finite-dimensional vector spaces S, T over $\mathbf{R}$, that (ii) no longer holds when the hypothesis on F is dropped.

[1] From Donne's Meditation XVII, which more famously is the source of "no man is an island" and "never send to know for whom the bell tools; it tolls for thee". The Carlyle quote is from *Sartor Resartus*, according to several sources on the Web. No, this has no direct connection with linear algebra.

5. Let V be a finite-dimensional vector space over an algebraically closed field F, and fix $A, B \in \mathrm{End}(V)$. Consider the linear operator $T = T_{A,B} : X \mapsto AX + XB$ on $\mathrm{End}(V)$.
   i) Express T in terms of tensor products (via the identification of $\mathrm{End}(V)$ with $V^* \otimes V$).
   ii) Describe the eigenvalues of T in terms of the eigenvalues of A and B.
   iii) Prove that if $F = \mathbf{C}$ and all eigenvalues of A, B has positive real part then every $M \in \mathrm{End}(V)$ can be written uniquely as $AX + XB$ for some $X \in \mathrm{End}(V)$.

Apropos eigenstuff… The next result generalizes what we saw about involutions (which are the special case $m = 2$, $\lambda_i = \pm 1$ over a field not of characterstic 2).

6. Suppose V is a vector space over a field F and T is a linear operator on V such that $\prod_{i=1}^{m}(T - \lambda_i I) = 0$ for some *distinct* $\lambda_i \in F$. Prove that V is the direct sum of the λ_i-eigenspaces of T. [NB: V may <u>not</u> be assumed finite-dimensional.]

Tensor products of A-modules. Like direct sums, quotient spaces, and duals, tensor products can be defined in the same way for modules over rings A that need not be fields. Basic properties such as $M \otimes (N \oplus N') \cong (M \otimes N) \oplus (M \otimes N')$ and $A \otimes M = M$ hold in this more general setting, and for much the same reason; but some new phenomena emerge, as in parts (ii) and (iii) of the next problem:

7. i) Show that if A is a commutative ring with unit, and $I \subseteq A$ is an ideal (an additive subgroup such that $aI \subseteq I$ for all $a \in A$, or equivalently a submodule of the A-module A), then the tensor product $(A/I) \otimes_A (A/I)$ of the quotient A-module A/I with itself is isomorphic with A/I.
   ii) On the other hand, show that $(\mathbf{Z}/2\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{Z}/3\mathbf{Z})$ is the trivial $\mathbf{Z}$-module $\{0\}$.
   iii) For positive integers m, n, what is the $\mathbf{Z}$-module $(\mathbf{Z}/m\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{Z}/n\mathbf{Z})$?

Finally, a bit about inner products:

8. Solve Exercise 20 on page 177, and 2 on page 189, of Axler (6A and 6B respectively). For #2, V is either a real or complex inner-product space, which need not be finite dimensional.

9. Is the symmetric bilinear pairing constructed in Problem 2 nondegenerate? When $F = \mathbf{R}$, is it positive definite?

Axler's exercise #20, as well as the more familiar #19 on the same page, is often referred to as the "polarization identity". They imply that a linear transformation preserves the norm if and only if it preserves the inner product (more precisely, they imply the harder, "only if" part of this result). These are basically also the identities used to prove 7.14 and 7.16 in the next chapter (pages 210, 211).

This problem set is due Wednesday, 18 October, in class.