**Math 55a: Honors Abstract Algebra**

Homework Assignment #5 (2 October 2017):
Linear Algebra V: "Eigenstuff"
(with a prelude on exact sequences and more duality)

> *The terms "proper value", "characteristic value", "secular value", and "latent-value" or "latent root" are sometimes used [for "eigenvalue"] by other authors. The latter term is due to Sylvester [Collected Papers III, 562–4] because such numbers are "latent in a somewhat similar sense as vapour may be said to be latent in water or smoke in a tobacco-leaf." We will not adhere to his terminology.*
>
> — N. Dunford and J.T. Schwartz: *Linear Operators, Part I*, pages 606–7.

A bit about exact sequences:

1. i) Suppose $0 \to V_1 \to V_2 \to V_3 \to \cdots \to V_n \to 0$ is an exact sequence of linear transformations between vector spaces all of which are finite dimensional. Prove that $\sum_{i=1}^n (-1)^i \dim V_i = 0$.
ii) Given positive integers $d_i$ $(i = 1, \ldots, n)$ such that $\sum_{i=1}^n (-1)^i d_i = 0$, must there exist an exact sequence as in (i) such that $\dim V_i = d_i$ for each $i$?

The next two questions explore further aspects of duality. For problem 2, vectors $v_1, \ldots, v_N$ in an $n$-dimensional vector space $V$ are said to be "in general linear position" if *every* choice of $n$ vectors $v_{i_1}, \ldots, v_{i_n}$ with $i_1 < i_2 < \cdots < i_N$ yields a basis for $V$. For example, this condition is satisfied by $v_i = (1, x_i) \in F^2$ for any pairwise distinct $x_i \in F$ (even though they are quite special in that any three points are collinear). More generally $(1, x_i, x_i^2, \ldots, x_i^d)$ works in $F^{d+1}$, again assuming the $x_i$ are pairwise distinct.

2. Let $V$ be an $n$-dimensional space over any field $F$, and for some $N \geq n$ let $v_1, \ldots, v_N \in V$ be any vectors that span $V$. Then we have a map $s : F^N \to V$ taking any $(a_1, \ldots, a_N)$ to $\sum_{i=1}^N a_i v_i$. By hypothesis $s$ is surjective. Hence we have an injective map $s^* : V^* \to (F^N)^*$. We've identified $F^N$ with its own dual, so we can regard $s^*$ as a map $V^* \to F^N$, and we then have a quotient map $q : F^N \to F^N/V^* =: W$, with $\dim W = N - n$. Let $w_1, \ldots, w_N \in W$ be the images of the unit vectors. Prove that $v_1, \ldots, v_N$ are in general linear position if and only if $w_1, \ldots, w_N$ are in general linear position.

3. Let $F$ be a field of characteristic zero, so $F$ contains a copy of $\mathbf{Z}$ with the same $0, 1, \pm, \times$. For a finite-dimensional vector space $V/F$, a "lattice" $L \subset V$ is the $\mathbf{Z}$-span of an $F$-basis for $V$, that is, an additive subgroup of the form

$$L = \left\{ \sum_{i=1}^n a_i v_i \mid a_i \in \mathbf{Z} \ (1 \leq i \leq n) \right\}$$

where $(v_1, \ldots, v_n)$ is a basis for $V$ (equivalently, the image of $\mathbf{Z}^n \subset F^n$ under an invertible linear map $F^n \to V$).[1] The *dual lattice* is a subset of the dual vector space $V^*$ defined by

$$L^* = \{v^* \in V^* \mid \forall v \in L,\ v^*(v) \in \mathbf{Z}\}.$$

Prove that $L^*$ is in fact a lattice in $V^*$.

The rest of the problems are taken from (or based on problems from) Chapter 5 of Axler. Unless stated otherwise $\mathbf{F}$ can be any field, and $\mathbf{C}$ can be any algebraically closed field; do not assume that vector spaces are finite-dimensional unless you must. From 5A:

4. Solve problems 2 and 3 (pages 138 and 139; remember that Axler's "null" and "range" are our "ker" and "image" respectively).

5. i) (Basically problem 13 on p.139) If $V$ is a finite-dimensional vector space over $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$, and $\epsilon$ is any positive real number, prove that for every $T \in \mathrm{End}(V)$ there exists $\alpha \in \mathbf{R}$ such that $|\alpha| < \epsilon$ and $T - \alpha I$ is invertible. (This is one way to show that when $V$ is finite-dimensional the invertible operators are "dense in $\mathrm{End}(V)$", using terminology that we'll develop at the start of 55b.)
   ii) Show (by constructing $V$ and $T$) that for any field $F$ there is a vector space $V/F$ and a linear operator $T : V \to V$ such that for all $\alpha \in F$ the operator $T - \alpha I$ is not invertible.

6.–7. Solve problems 15 and 21 on page 140.

8.–9. Solve problems 29 and 31 on page "$\approx 100\sqrt{2}$"; for Problem 31, assume that $\mathbf{F}$ has at least $m$ elements, which is clearly necessary. (Hint for that problem: you can replace the assumption that $V$ has finite dimension by the hypothesis that every finite-dimensional subspace of $V$ has a complement.)

From 5B:

10. Solve exercises 5 and 10 on page 153. (Naturally exercise 5 here is related with 5A exercise 15.)

11. Solve exercises 11, 12 on page 153. For 12, only one of "if" and "only if" fails over $\mathbf{R}$ — which one? — and the other holds over any field.

Exercise 10 has the following important consequence: if $P \in F[z]$ and $P(T) = 0$ for some linear operator $T \in \mathrm{End}(V)$, then every eigenvalue of $T$ is a root of $P$. For instance, the only possible eigenvalues of a linear involution are $\pm 1$, the roots of $z^2 - 1$. Several other exercises on this page are variations on this theme.

This problem set is due *Wednesday*, 11 October,[2] at the beginning of class.

[1] NB once $n \geq 2$ a lattice, like a vector space, can have many different choices of generators $v_i$; e.g. $\mathbf{Z}^2$ itself we can choose $v_1 = (20, 17)$ and $v_2 = (7, 6)$. We shall pursue this further after developing the determinant and related constructions.

[2] because Monday the 9th is a University holiday.