Math 55a: Honors Abstract Algebra

Homework Assignment #5 (30 Septemer 2016):
Linear Algebra V: "Eigenstuff"
(with a prelude on exact sequences and more duality)

The terms "proper value", "characteristic value", "secular value", and "latent-value" or "latent root" are sometimes used [for "eigenvalue"] by other authors. The latter term is due to Sylvester [Collected Papers III, 562–4] because such numbers are "latent in a somewhat similar sense as vapour may be said to be latent in water or smoke in a tobacco-leaf." We will not adhere to his terminology.

— N. Dunford and J.T. Schwartz: *Linear Operators, Part I*, pages 606–7.

A bit about exact sequences:

1. i) Suppose $0 \to V_1 \to V_2 \to V_3 \to \cdots \to V_n \to 0$ is an exact sequence of linear transformations between vector spaces all of which are finite dimensional. Prove that $\sum_{i=1}^n (-1)^i \dim V_i = 0$.
   ii) Given positive integers d_i $(i = 1, \ldots, n)$ such that $\sum_{i=1}^n (-1)^i d_i = 0$, must there exist an exact sequence as in (i) such that $\dim V_i = d_i$ for each i?

The next two questions explore further aspects of duality. For problem 2, vectors $v_1, \ldots, v_N$ in an n-dimensional vector space V are said to be "in general linear position" if *every* choice of n vectors $v_{i_1}, \ldots, v_{i_n}$ with $i_1 < i_2 < \cdots < i_N$ yields a basis for V. For example, this condition is satisfied by $v_i = (1, x_i) \in F^2$ for any pairwise distinct $x_i \in F$ (even though they are quite special in that any three points are collinear). More generally $(1, x_i, x_i^2, \ldots, x_i^d)$ works in F^{d+1}, again assuming the x_i are pairwise distinct.

2. Let V be an n-dimensional space over any field F, and for some $N \geq n$ let $v_1, \ldots, v_N \in V$ be any vectors that span V. Then we have a map $s : F^N \to V$ taking any $(a_1, \ldots, a_N)$ to $\sum_{i=1}^N a_i v_i$. By hypothesis s is surjective. Hence we have an injective map $s^* : V^* \to (F^N)^*$. We've identified F^N with its own dual, so we can regard s^* as a map $V^* \to F^N$, and we then have a quotient map $q : F^N \to F^N/V^* =: W$, with $\dim W = N - n$. Let $w_1, \ldots, w_N \in W$ be the images of the unit vectors. Prove that $v_1, \ldots, v_N$ are in general linear position if and only if $w_1, \ldots, w_N$ are in general linear position.

3. Let F be a field of characteristic zero, so F contains a copy of $\mathbf{Z}$. For a finite-dimensional vector space V/F, a "lattice" $L \subset V$ is the $\mathbf{Z}$-span of an F-basis for V, that is, an additive subgroup of the form

$$L = \left\{ \sum_{i=1}^n a_i v_i \mid a_i \in \mathbf{Z} \ (1 \leq i \leq n) \right\}$$

where $(v_1, \ldots, v_n)$ is a basis for V (equivalently, the image of $\mathbf{Z}^n \subset F^n$ under an invertible linear map $F^n \to V$).[1] The *dual lattice* is a subset of the dual vector space V^* defined by

$$L^* = \{v^* \in V^* \mid \forall v \in L,\ v^*(v) \in \mathbf{Z}\}.$$

Prove that L^* is in fact a lattice in V^*.

The rest of the problems are taken from (or based on problems from) Chapter 5 of Axler. Unless stated otherwise $\mathbf{F}$ can be any field, and $\mathbf{C}$ can be any algebraically closed field; do not assume that vector spaces are finite-dimensional unless you must. From 5A:

4. Solve problems 2 and 3 (pages 138 and 139; remember that Axler's "null" is our "ker").

5. (Basically problem 13 on p.139)
i) If V is a finite-dimensional vector space over $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$, and ϵ is any positive real number, prove that for every $T \in \mathrm{End}(V)$ there exists $\alpha \in \mathbf{R}$ such that $|\alpha| < \epsilon$ and $T - \alpha I$ is invertible. (This is one way to show that when V is finite-dimensional the invertible operators are "dense in $\mathrm{End}(V)$", using terminology that we'll develop at the start of 55b.)
ii) Show (by constructing V and T) that for any field F there is a vector space V/F and a linear operator $T : V \to V$ such that for all $\alpha \in F$ the operator $T - \alpha I$ is not invertible.

6.–7. Solve problems 15 and 21 on page 140.

8.–9. Solve problems 29 and 31 on page "$\approx 100\sqrt{2}$" (hint for Problem 31: you can replace the assumption that V has finite dimension by the hypothesis that every finite-dimensional subspace of V has a complement.)

From 5B:

10. Solve exercises 5 and 10 on page 153. (Naturally this is related with 5A exercise 15.)

11. Solve exercises 11, 12 on page 153. For 12, only one of "if" and "only if" fails over $\mathbf{R}$ — which one? — and the other holds over any field.

Exercise 10 has the following important consequence: if $P \in F[z]$ and $P(T) = 0$ for some linear operator $T \in \mathrm{End}(V)$, then every eigenvalue of T is a root of P. For instance, the only possible eigenvalues of a linear involution are ± 1, the roots of $z^2 - 1$. Several other exercises on this page are variations on this theme.

This problem set is due Friday, 7 October, at the beginning of class.

[1] NB once $n \geq 2$ a lattice, like a vector space, can have many different choices of generators v_i; e.g. $\mathbf{Z}^2$ itself we can choose $v_1 = (20, 17)$ and $v_2 = (7, 6)$. We shall pursue this further after developing the determinant and related constructions.