Math 55a: Honors Abstract Algebra

Homework Assignment #2 (9 September 2016):
Linear Algebra II

> **TFAE (The Following Are Equivalent)**: If I say this it means that, and if I say that it means the other thing, and if I say the other thing...[1]

This homework assignment consists of four problems on the basic definitions and properties of vector spaces and their dimensions, four problems showing these ideas in different contexts, and a bit more about modules. Problem set is due Friday, Sep. 16 in class.

1. Solve Exercise 2.A 14 on page 38 of the Axler textbook for vector spaces over an arbitrary field $\mathbf{F}$; deduce Exercises 15 and 16. ["$\mathbf{F}^\infty$" is the vector space of Example 1.22 on page 13 (also described in class Wednesday Sep.7); NB this is $\prod_{j=1}^\infty \mathbf{F}$, not $\oplus_{j=1}^\infty \mathbf{F}$.]

2. Solve Exercises 2.C 14, 16, and 17 on page 49. (For 17, cf. 2.43 on page 47.)

3. What is the dimension of the vector space generated by...
   i) The vectors $(1, i, 1+i, 0)$, $(-i, 1, 1-i, 0)$, and $(1-i, 1+i, 2, 0)$ in $\mathbf{C}^4$?
   ii) The functions $f(x) = \sin(x)$, $\sin(x + \pi/6)$, and $\sin(x + \pi/3)$ in $\mathbf{R}^\mathbf{R}$?
   iii) The n vectors
   $$\begin{array}{l} (1,1,0,0,0,\ldots,0,0,0), \\ (0,1,1,0,0,\ldots,0,0,0), \\ (0,0,1,1,0,\ldots,0,0,0), \\ \qquad\qquad\vdots \\ (0,0,0,0,0,\ldots,1,1,0), \\ (0,0,0,0,0,\ldots,0,1,1), \\ (1,0,0,0,0,\ldots,0,0,1) \end{array}$$
   in $\mathbf{R}^n$ $(n \geq 2)$? [Note the first coordinate of the last vector!]

4. Find a basis for:
   i) The subspace $\{\vec{x} : x_1 + \cdots + x_n = 0\}$ of F^n (this is the space we called "F_0^n").
   ii) The subspace $\{P : P(-3) = 0\}$ of $\mathcal{P}$.
   (In each case there are many right answers — but even more wrong ones.)

5. (Polynomial interpolation.) Let $a_1, a_2, \ldots, a_n$ be (pairwise) distinct elements of a field F. Prove that the following are equivalent:
   i) For any $p_1, p_2, \ldots, p_n \in F$ there exists a unique polynomial $P(x)$ of degree less than n such that $P(a_i) = p_i$ for each $i = 1, 2, \ldots, n$.
   ii) The n vectors $v_i := (a_1^i, a_2^i, \ldots, a_n^i)$ $(0 \leq i < n)$ in F^n are linearly independent.
   Then prove one (and thus both) of those statements. [*Do not use determinants* even if you have seen them already!]

6. (Complexification of a real vector space.) Prove that for any $\mathbf{R}$-vector space V one may regard $V \oplus V$ as a $\mathbf{C}$-vector space $V_\mathbf{C}$ by defining the scalar multiplication by
$$(a + ib)(v, w) = (av - bw, aw + bv)$$

[1] *Definitions of Terms Commonly Used in Higher Math*, R. Glover et al.

for all $a, b \in \mathbf{R}$ and $v, w \in V$. [We shall later see that this is an example of a "tensor product", namely $V_{\mathbf{C}} = V \otimes_{\mathbf{R}} \mathbf{C}$.] If V is finite dimensional, what is the dimension of $V_{\mathbf{C}}$ as a $\mathbf{C}$-vector space?

7. Let V be a vector space over a field F, and let K a subfield of F (i.e., a subset containing $0, 1$ which is closed under $\pm$, $\times$, and multiplicative inverse, and thus constitutes a field with the arithmetic operations defined by restriction from K). Note that V and F may also be regarded as K-vector spaces by restricting the arithmetic operations appropriately. (For instance, $\mathbf{C}$ is an $\mathbf{R}$-vector space of dimension 2, and any $\mathbf{C}$-vector space is automatically an $\mathbf{R}$-vector space as well.) Show that if $m = \dim_K(F)$ and $n = \dim_F(V)$ are finite, then so is $d = \dim_K(V)$, and express d in terms of m and n.

8. Let F be a finite field, and let q be the number of elements of F. For some positive integer n consider the F-vector space $V = F^n$.
   i) Prove that V has q^n elements and
   $$\prod_{i=0}^{n-1}(q^n - q^i) = (q^n - 1)(q^n - q)(q^n - q^2)\cdots(q^n - q^{n-1})$$
   ordered bases $(e_1, e_2, \ldots, e_n)$.
   ii) Now let k be an integer such that $0 \le k \le n$. How many subspaces of dimension k does V have?

We noted that the basic property "$av = \vec{0} \Rightarrow a = 0$ or $v = \vec{0}$" of vector spaces fails for modules. But often in mathematics such "failures" are also opportunities ...
Suppose M is a module over $\mathbf{Z}$, which is to say an abelian group. For an integer $n \neq 0$, we say $x \in M$ is an <u>n-torsion element</u> if $nx = 0$, and a <u>torsion element</u> if it is n-torsion for some n. Let $M[n]$ be the set of n-torsion elements, and $M_{\text{tors}} = \cup_{n \neq 0} M[n]$ the set of torsion elements. These are called the "n-torsion subgroup" and "torsion subgroup" of M, a terminology justified by the first part of the next problem.

9. i) Prove that $M[n]$ is a subgroup of M for each n, and that M_{tors} is a subgroup of M.
   ii) How might (i) fail if we use the same definitions of $M[n]$ and M_{tors} for a module M over an arbitrary commutative ring A (with n now allowed to be any nonzero element of A)? How can you fix the definitions so (i) works (and still gives something new — a "torsion submodule") in this generality?

10. An abelian group M is said to be *divisible* if for every nonzero integer n the map $M \to M$, $x \mapsto nx$ is surjective (i.e. the equation $nx = b$ has a solution $x \in M$ for all $b \in M$).
   i) Give an example of a divisible group $M \neq \{0\}$ whose only torsion element is 0.
   ii) Give an example of a divisible group $M \neq \{0\}$ such that $M = M_{\text{tors}}$. (A group M satisfying $M = M_{\text{tors}}$ is said to be a "torsion group".)