Math 55a: Honors Abstract Algebra

Homework Assignment #11 (21 November 2016):
Representations of finite groups

$$i^2 = j^2 = k^2 = ijk = -1$$

—W.R. Hamilton, 1843 [cut into a stone on Brougham[1] Bridge, Dublin; see also the final two problems].

We start with some applications of the general theory to permutation representations. Recall that if a finite group G acts on a finite set S then $\mathbf{C}^S$ is a representation of G and the associated character takes any $g \in G$ to the number of fixed points of g.

1. Let $V = \mathbf{C}^S$. Prove that the dimension of the fixed subspace V^G is the number of orbits of the action of G on S, both by identifying V^G explicitly in terms of the orbit decomposition and by using the formula $\langle \chi, 1 \rangle$ for that dimension. Deduce that G is transitive <u>iff</u> $\dim V^G = 1$ <u>iff</u> $\langle \chi, 1 \rangle = 1$.

2. Suppose then that G acts transitively on S. Let V_0 be the orthogonal complement "$V \ominus V^G$" of V^G in V, and let χ_0 be its character. Determine $\langle \chi_0, \chi_0 \rangle$, and deduce that V_0 is irreducible if and only if G acts doubly transitively on S. [A group action on S is said to be "doubly transitive" when it is transitive on ordered pairs (s_1, s_2) with $s_1, s_2 \in S$ and $s_1 \neq s_2$.]

3. i) Show that for $k = 1, 2, 3, \ldots$ the permutation representation of G on S^k is isomorphic with $V^{\otimes k}$. Deduce a formula for the number of G-orbits on S^k. (The action of G on S is not required to be transitive, and the action on S^k is coordinatewise.)
ii) Give a similar formula for the number of G-orbits on the set k^S of k-colorings of S.
iii) Use this formula to show that there are 36 carbon tetrahalides. (A "carbon tetrahalide" is a molecule CX_4 where each X is one of the four halogens F, Cl, Br, I, and the four X's are vertices of a tetrahedron centered on the C atom.) Verify this count directly. [Note that there are two kinds of CFClBrI because only orientation-preserving symmetries are allowed, so an asymmetric molecule is distinct from its mirror image (chemists call such mirror images "enantiomers"); that is, the relevant group G is what Artin calls the tetrahedral group T of order 12.]

What about the character of $(\mathrm{Sym}^k V, \mathrm{Sym}^k \rho)$ or $(\bigwedge^k V, \bigwedge^k \rho)$? Here the formulas are more complicated; the character of $g \in G$ depends on the full characteristic polynomial of $\rho(g)$, not just its trace. It's easier to formulate the result in terms of a generating function, which is a formal power series $X(g) = \sum_{k=0}^{\infty} \chi_k(g) T^k$ where $\chi_k(g)$ is the trace of $(\mathrm{Sym}^k \rho)(g)$ or $(\bigwedge^k \rho)(g)$ respectively.

4. i) For $\bigwedge^k$ this generating function is a polynomial of degree $\dim(V)$, because once $k > \dim(V)$ the k-th exterior power is the zero space so the trace is zero as well. Show that this polynomial is the determinant of $1 + T\rho(g)$.
ii) For Sym^k, show that $X(g) = 1 / \det(1 - T\rho(g))$.
iii) Now let $G = S_3$ and V be the 3-dimensional permutation representation. Thus S_3 acts on the polynomial ring $\mathbf{C}[z_1, z_2, z_3] = \bigoplus_{k=0}^{\infty} \mathrm{Sym}^k V$ by permuting the variables

[1] a.k.a. Broom, which sounds the same in one pronunciation of "brougham", which is a kind of horse-drawn carriage — and the source of "broughammed" ("traveled by brougham", or possibly "equipped with a brougham") which is a candidate for the longest English-language monosyllable. But I digress.

z_1, z_2, z_3. Show that $\dim((\mathrm{Sym}^k V)^G)$ is the X^k coefficient of the generating function $((1-X)(1-X^2)(1-X^3))^{-1}$, and explain why this is consistent with the known result that the subring of $\mathbf{C}[z_1, z_2, z_3]$ invariant under the action of S_3 consists of polynomials in the elementary symmetric functions $z_1 + z_2 + z_3$, $z_2z_3 + z_3z_1 + z_1z_2$, and $z_1z_2z_3$ of degrees $1, 2, 3$.

The irreducible representations of the direct product of two finite groups:

5. i) Let (V_1, ρ_1) and (V_2, ρ_2) be complex representations of finite groups G_1, G_2. Define a representation of (V, ρ) of $G := G_1 \times G_2$ by $V = V_1 \otimes V_2$ and $\rho((g_1, g_2)) = \rho_1(g_1) \otimes \rho_2(g_2)$ for all $g_1 \in V_1$ and $g_2 \in V_2$. Find the character of (V, ρ), and deduce that (V, ρ) is irreducible if and only if both (V_1, ρ_1) and (V_2, ρ_2) are irreducible.
   ii) Prove that every irreducible representation of G arises from the construction in part (i) for some irreducible representations $V_1 = (V_1, \rho_1)$ and $V_2 = (V_2, \rho_2)$. [Hint: first show that if V_1, V_2 are irreducible then $V_1 \otimes V_2$ cannot be isomorphic with $W_1 \otimes W_2$ unless $V_1 \cong W_1$ and $V_2 \cong W_2$.]

The *Hamilton quaternions* are the skew field $\mathbf{H}$ defined as follows: $\mathbf{H}$ is a 4-dimensional algebra over $\mathbf{R}$ with basis $1, i, j, k$ and multiplication characterized by the properties that where 1 is the multiplicative identity while $i^2 = j^2 = k^2 = ijk = -1$ (so for instance $ij = k = -ji$). The *quaternion group* Q_8 is the subgroup $\{\pm 1, \pm i, \pm j, \pm k\}$ of $\mathbf{H}^*$. In the last two problems you'll verify that $\mathbf{H}$ is indeed a skew field and construct a representation W of Q_8 over $\mathbf{R}$ that is irreducible over $\mathbf{R}$ but not over $\mathbf{C}$ and has $\mathrm{End}_G(W) \cong \mathbf{H}$.

6. i) Let $\mathcal{A}$ be the $\mathbf{R}$-vector space of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, b, c, d \in \mathbf{C}$ such that $\bar{a} = d$ and $\bar{b} = -c$. Prove that $\mathcal{A}$ is closed under matrix multiplication, and every nonzero $A \in \mathcal{A}$ is invertible.
   ii) For $A \in \mathcal{A}$ define $\sigma(A) = \mathrm{tr}(A)I - A$. Prove that σ is an anti-involution of $\mathcal{A}$ (that is, σ is a vector space involution of $\mathcal{A}$ satisfying $\sigma(AA') = \sigma(A')\sigma(A)$ for all $A, A' \in \mathcal{A}$). Compute $A\sigma(A)$ and $\sigma(A)A$, and use this to prove that $\mathcal{A}$ contains the inverse of every nonzero $A \in \mathcal{A}$.
   iii) Find an isomorphism between $\mathbf{H}$ and $\mathcal{A}$ that identifies σ with the anti-involution taking $1, i, j, k$ to $+1, -i, -j, -k$ respectively. (This anti-involution is called "conjugation" in $\mathbf{H}$, and denoted by $q \leftrightarrow \bar{q}$ as is done for complex conjugation.)

7. i) The identification of $\mathbf{H}$ with $\mathcal{A}$ yields a 2-dimensional complex representation V of $\mathbf{H}^*$, and thus of Q_8. Prove that the character of any $q \in \mathbf{H}^*$ is the real number $q + \bar{q}$. Deduce that V is an irreducible representation of Q_8. [You'll recognize its character if you went to section last week.]
   ii) The action of $\mathbf{H}^*$ on $\mathbf{H}$ by multiplication from the left gives $\mathbf{H}$ the structure of a 4-dimensional real representation W of $\mathbf{H}^*$, and thus of Q_8. Compute its character for any $q \in \mathbf{H}^*$, and verify that it equals $2\chi_V(q)$. On the other hand, multiplication *from the right* by any $q \in \mathbf{H}$ commutes with our action, and this shows that $\mathrm{End}_{Q_8}(W)$ contains a copy of $\mathbf{H}$. Prove that in fact $\mathrm{End}_{Q_8}(W) \cong \mathbf{H}$.
   iii) Use this to show that W is irreducible as a real representation of Q_8.

[Another route to the result of (iii) starts by using the general theory to find that $W \otimes_R \mathbf{C}$ is isomorphic with $V \oplus V$ as a representation of Q_8; thus if W were reducible it would be a direct sum of two irreducible real representations of Q_8, with $-1 \in Q_8$ acting on both by multiplication by -1. But, as with the "unitary trick" for complex representations, any real representation of a finite group has an invariant orthogonal form, so we'd get a homomorphism $Q_8 \to \mathrm{O}_2(\mathbf{R})$ taking -1 to -1, and this is soon seen to be impossible, e.g. using the facts that $\mathrm{SO}_2(\mathbf{R})$ is commutative and each element of $\mathrm{O}_2(\mathbf{R})$ that is not in $\mathrm{SO}_2(\mathbf{R})$ is an involution.]

This final problem set is due *Wednesday*, November 30 at 5 P.M.