Math 55a: Honors Abstract Algebra

Homework Assignment #9 (1 November 2010):
Linear Algebra IX: Exterior algebra and determinants, cont'd

> $\ast$, 231
> —Axler, Symbol Index on p.247
>
> The symbols $\ast$ that appear on the first page of these chapters are decorations [...] Chapter 1 has one of these symbols, Chapter 2 has two of them, and so on. The symbols get smaller with each chapter. What you may not have noticed is that the sum of the areas of the symbols at the beginning of each chapter is the same [...]
> —Axler, p.231 [answering a question posed in PS1]

A couple of Axler problems to start with:

1. i) Solve Problem 9 on page 244 of the textbook. What is the trace of P?
   ii) Solve Problem 22 on page 246. This should be easy to do with the $\bigwedge^n$ definition of the determinant, even though Axler didn't intend that.

Symmetric pairings from exterior algebra:

2. Let V be a vector space of dimension 4 over any field F not of characteristic 2.[1] Let $W = \bigwedge^2 V$, and fix a nonzero $\psi \in \bigwedge^4 V$. We have seen that there is a symmetric nondegenerate pairing $W \times W \to F$ defined by $\omega \wedge \omega' = \langle \omega, \omega' \rangle \psi$. Prove that $\langle \omega, \omega \rangle = 0$ if and only if $\omega = v_1 \wedge v_2$ for some $v_1, v_2 \in V$.

3. Now let V be a vector space of dimension $4n$ over $\mathbf{R}$ for some positive integer n. Let $W = \bigwedge^{2n} V$, and fix a nonzero $\psi \in \bigwedge^{4n} V$. Again there is a symmetric nondegenerate pairing $W \times W \to F$ defined by $\omega \wedge \omega' = \langle \omega, \omega' \rangle \psi$. Since W is now a real vector space, this pairing has a signature. Find it, and construct a decomposition $W = W_+ \oplus W_-$ into orthogonal subspaces such that the pairing is positive-definite on W_+ and negative-definite on W_-.

The determinant and other properties of a circulant matrix:

4. An $n \times n$ matrix A is said to be *circulant* when its (i, j)-th entry a_{ij} depends only on $i - j \bmod n$. Let S be the circulant matrix for which a_{ij} is 1 if $j \equiv i + 1 \bmod n$ and 0 otherwise. Show that A is circulant if and only if A is a polynomial in S. Conclude that all circulant matrices commute and are normal. What are the eigenvalues, eigenvectors, and determinant of a circulant matrix with entries in $\mathbf{C}$? [This can be viewed as part of a theory of discrete Fourier analysis, or of representations of the cyclic group $\mathbf{Z}/n\mathbf{Z}$. The case $n = 2$ of the determinant formula is well known; less so the $n = 3$ determinant, though it is still memorable, and shows up every once in a while — usually in contexts less frivolous than my observation about 27 years ago that $1983 \mid 2^{33} - 5 \cdot 2^{17} - 1$.]

Linear algebra as a tool for studying field algebra, continued: Let F be any field, $\mathcal{P}$ the polynomial ring $F[z]$, and $\mathcal{P}_k \subset \mathcal{P}$ ($k = 0, 1, 2, \ldots$) the F-vector space of polynomials of degree at most k.

[1] The construction here can be modified to work in characteristic 2 as well; the result stays the same.

5. i) Fix $P, Q \in \mathcal{P}$ of positive degree m, n.[2] For any k, l, determine the dimension of the subspace of $\mathcal{P}_k \oplus \mathcal{P}_l$ consisting of pairs (X, Y) such that $PX + QY = 0$.
   ii) Use this to construct a square matrix $M = M_{P,Q}$ of size $m + n$ that is nonsingular if and only if P, Q have no common factor, and determine the rank of M for any P, Q. Your $M_{P,Q}$ should be the image of (P, Q) under a *linear* map from $\mathcal{P}_m \oplus \mathcal{P}_n$ to $\mathrm{End}(F^{m+n})$.
   iii) Your analysis in (i) should give you either the row or the column nullspace of M (depending on how you set it up). Describe the other nullspace, at least in the case that $\gcd(P, Q)$ has degree 1.

6. Fix $p \in \mathcal{P}$ of degree $n > 0$. Let $V = \mathcal{P}/p\mathcal{P}$, an n-dimensional vector space over F (which also inherits a ring structure from $\mathcal{P}$), and $T : V \to V$ the operator taking any equivalence class $[Q]$ to $[zQ]$.
   i) Determine the minimal and characteristic polynomials of T.
   ii) Assume that p is irreducible. Let $\alpha \in \mathcal{P}$ be a polynomial not in $p\mathcal{P}$. Prove that the operator on V defined by $Q \mapsto \alpha Q$ is injective, and thus invertible. Conclude that V is a field. (The fact that V is *not* a field if p is reducible is easy, as observed in class some time ago for the analogous case of $\mathbf{Z}/n\mathbf{Z}$.)

The trace and determinant of the multiplication-by-$[\alpha]$ map on V are called the *trace* and *norm* of $[\alpha]$. It's easy to see that these are respectively an F-linear functional on V and a multiplicative map from V to F.

Determinants and inner products (and another application of Gram-Schmidt):

7. i) Let $F = \mathbf{R}$ or $\mathbf{C}$, and $v_1, v_2, \ldots, v_n \in F^n$ the row vectors of an $n \times n$ matrix A. Prove that

$$|\det A| \leq \prod_{i=1}^{n} \|v_i\|$$

   where $\|\cdot\|$ is the usual norm on F^n, with equality if and only if the v_i are orthogonal with respect to the corresponding inner product.
   ii) Deduce that if M is a positive-definite symmetric or Hermitian $n \times n$ matrix with entries $a_{i,j}$ then

$$\det M \leq \prod_{i=1}^{n} a_{i,i},$$

   with equality if and only if M is diagonal.

(We know already that $\det M$ and the diagonal entries $a_{i,i}$ are positive real numbers.)

And pfinally: A square matrix A with entries a_{ij} in a field F is said to be *skew-symmetric* if its entries satisfy $a_{ij} = -a_{ji}$ for all i, j and the diagonal entries a_{ii} all vanish. We'll show in class that $\det A = 0$ if A has odd order; here we study the even case.

8. If A has even order $2n$, and $n!$ is invertible in F, the *Pfaffian* $\mathrm{Pf}(A)$ can be defined thus: let $\omega \in \wedge^2(F^{2n})$ be defined by $\omega = \sum_{1 \leq i < j \leq 2n} a_{ij}\, e_i \wedge e_j$; then $\mathrm{Pf}(A) \in F$ is the scalar such that

$$\omega^n = n!\, \mathrm{Pf}(A)(e_1 \wedge e_2 \wedge \cdots \wedge e_{2n})$$

   in $\wedge^{2n}(F^{2n})$. (Of course ω^n means $\omega \wedge \omega \wedge \cdots \wedge \omega$ with n factors.) Give an explicit formula for $\mathrm{Pf}(A)$ in terms of the a_{ij}, analogous to the formula for the determinant as a sum of $n!$ monomials. Prove that

$$\det(A) = (\mathrm{Pf}(A))^2.$$

This problem set is due *Wednesday*, 10 November, at the beginning of class.

[2] I mean this literally: P and Q should have nonzero coefficients of z^m and z^n respectively.