Math 55a: Honors Abstract Algebra

Homework Assignment #8 (25 October 2010):
Linear Algebra VIII:

As soon as I get into [Math 55] class, I'm fighting off a swarm
Of positive-definite non-degenerate symmetric bilinear forms!
—from a somewhat redundantly titled patter-song in *Les Phys* (P.Dong, 2001)

(In general, PDNDSBF's are probably easier to compute with than determinants and the like, but it's harder to fit "extrerior algebra" into G&S-style lyrics. . .)

More about nilpotent operators etc.:

1.–2. [sic] Solve Exercises 4, 5, 8, and 15 in Axler Chapter 8 (pages 188 and 189). For problem 4, cf. Axler, 177–178, and Exercise 19 on page 190 [NB we're not covering those pages in class, and I'm not assigning Exercise 19.] For which of these Exercises can **C** be replaced by an arbitrary field, or an arbitary algebraically closed field?

3. An operator U on a finite-dimensional vector space is said to be "unipotent" if $U - I$ is nilpotent. (Over an algebraically closed field, this is equivalent to the condition that 1 is the only eigenvalue of U.) Thus Axler's Lemma 8.30 is the assertion that a unipotent operator always has a square root. He notes (p.178) that the proof, and thus the result, is valid equally over **C** and **R**. For which fields F does the proof not work? Prove that for every such field there is a unipotent operator on some finite-dimensional F-vector space that does not have a square root. Generalize to k-th roots for all $k = 2, 3, 4, \ldots$.

More about traces and such:

4. Let V be an inner-product space with orthonormal basis $(e_1, \ldots, e_n)$. For operators $S, T \in \mathcal{L}(V)$, define
$$\langle S, T\rangle := \sum_{j=1}^{n} \langle Se_j, Te_j\rangle.$$
Prove that $\langle\cdot,\cdot\rangle$ is an inner product on $\mathcal{L}(V)$, that it satisfies the identity $\langle S, T\rangle = \langle T^*, S^*\rangle$, and that the inner product does not depend on the choice of orthonormal basis for V. [Thus an inner product on finite-dimensional space V yields canonically an inner product on $\mathcal{L}(V)$. Cf. also Problem 2 on PS5.]

5. Solve Exercises 16 and 19 from Chapter 10 of the textbook (pages 245 and 246).

Exterior algebra, continued:

6. Fix a nonnegative integer k. Let $F = \mathbf{R}$ or $\mathbf{C}$, and let V/F be a vector space equipped with an inner product $\langle\cdot,\cdot\rangle$. For $v_1, \ldots, v_k, w_1, \ldots, w_k \in V$, define
$$\langle\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k\rangle\rangle$$
to be the determinant of the $k \times k$ matrix whose (i, j) entry is $\langle v_i, w_j\rangle$. Prove that $\langle\langle\cdot,\cdot\rangle\rangle$ extends to an inner product on the exterior power $\bigwedge^k V$.

7. Let V be a finite-dimensional vector space over a field F. For $\omega \in \bigwedge^2 V$, define the *rank* of ω to be the rank of the associated alternating pairing on V^*, which in turn is the rank of the corresponding map $V^* \to V$. [If $\omega = \sum_i a_i\,(v_i \wedge w_i)$, the pairing is given by

$$\langle v^*, w^* \rangle = \sum_i a_i\,(v^*(v_i)\,w^*(w_i) - v^*(w_i)\,w^*(v_i)).]$$

In Problem 7 of the last problem set we saw in effect that this rank is an even integer, say $2k$. Use the results of that problem to give the following characterizations of k:

i) For each $m = 0, 1, 2, \ldots$, there exist $u_i, v_i \in V$ $(i = 1, \ldots, m)$ such that

$$\omega = \sum_{i=1}^{m} u_i \wedge v_i$$

if and only if $m \geq k$. Moreover, if $m = k$ then the $2k$ vectors u_i, v_i are linearly independent.

ii) In characteristic 0 or $p > 2k$, the k-th exterior power of ω (that is, the element $\omega \wedge \omega \wedge \cdots \wedge \omega$ of $\bigwedge^{2k} V$, with k factors of ω) is nonzero, but the $(k+1)$-st exterior power vanishes.

Finally, we'll use the sign homomorphism (which we introduce to study exterior algebra) to prove familiar(?) facts about Rubik's Cube, and about a related 19th-century craze. Some terminology first: A permutation is called *even* or *odd* according as its sign is $+1$ or -1 respectively. If $i_1, i_2, \ldots, i_m$ are m distinct integers in $\{1, 2, \ldots, n\}$, the permutation of $\{1, 2, \ldots, n\}$ that takes i_1 to i_2, i_2 to i_3, $\ldots$, i_r to i_{r+1}, $\ldots$, i_{m-1} to i_m, and i_m back to i_1, while leaving the rest of $\{1, 2, \ldots, n\}$ fixed, is called an *m-cycle.* (In particular, the identity permutation is a 1-cycle.)

8. i) Prove that an m-cycle has sign $(-1)^{m+1}$, i.e., is even iff m is odd.

ii) Prove that no sequence of turns of Rubik's Cube can have the effect of flipping one of its edge pieces while leaving the rest unchanged.

iii) Prove that no sequence of turns of Rubik's Cube can have the effect of switching two of its edge pieces while leaving the rest unchanged. Does this approach work for the $4 \times 4 \times 4$ Cube?

9. The "15-puzzle"[1] is a sliding puzzle of 15 numbered 1×1 tiles confined to a 4×4 square. The initial configuration is

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Show that in any reachable configuration with the empty space in the bottom right corner the tiles have undergone an even permutation. [Hint: How many moves has the empty space made?] This necessary condition is sufficient, but I do not ask you to carry out the construction that proves this.

This problem set is due Monday, 1 November, at the beginning of class.

[1] Commonly associated with Sam Loyd, though probably not in fact one of the many puzzles he created.