Math 55a: Honors Abstract Algebra

Homework Assignment #5 (1 October 2010):
Linear Algebra V: tensors, more eigenstuff, and a bit on inner products

The terms "proper value", "characteristic value", "secular value", and "latent-value" or "latent root" are sometimes used [for "eigenvalue"] by other authors. The latter term is due to Sylvester [Collected Papers III, 562–4] because such numbers are "latent in a somewhat similar sense as vapour may be said to be latent in water or smoke in a tobacco-leaf." We will not adhere to his terminology.

— N. Dunford and J.T. Schwartz: *Linear Operators, Part I*, pages 606–7.

We begin with some basic problems on tensors and tensor products. For the first of these, recall that the "rank" of a linear transformation $T : U \to V$ is the dimension of its image $T(U)$; the *rank* of a matrix is the rank of the linear transformation it represents.

1. Let $\{u_i\}_{i=1}^m$ and $\{v_j\}_{j=1}^n$ be bases of the F-vector spaces U and V, and consider the general element $w = \sum_i \sum_j w_{ij}(u_i \otimes v_j)$ of $U \otimes V$. Prove that w is the sum of r pure tensors if and only if the matrix (w_{ij}) has rank at most r.

2. Let V be a vector space of finite dimension n over a field F. We constructed a linear map, the trace, from $\mathcal{L}(V)$ to F. Hence the map from $\mathcal{L}(V) \times \mathcal{L}(V)$ to F taking (S, T) to the trace of ST is bilinear. Prove that it is symmetric. For what n can there exist $S, T \in \mathcal{L}(V)$ such that $ST - TS$ is the identity map? (By comparison, observe that the operators $P \mapsto dP/dz$ and $P \mapsto zP$ on the infinite-dimensional space $\mathcal{P} = F[z]$ satisfy $ST - TS = I$.)

Tensors and eigenstuff:

3. Fix $a \in \mathbf{C}$, and let $T : \mathbf{C} \to \mathbf{C}$ be the map $z \mapsto az$. This is an $\mathbf{R}$-linear operator, so we may consider the linear operator $T' = T \otimes 1$ on the complex vector space $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$. What are the eigenvalues and eigenvectors of T'? (Warning: The answer depends on whether $a \in \mathbf{R}$.)

4. Let U, V be vector spaces over a field F, equipped with linear operators $S \in \mathcal{L}(U)$, $T \in \mathcal{L}(V)$. Consider $S \otimes T \in \mathcal{L}(U \otimes V)$.
   i) If $\lambda \in F$ is an eigenvalue of S, and $\mu \in F$ is an eigenvalue of T, prove that $\lambda\mu$ is an eigenvalue of $S \otimes T$.
   ii) If U, V are finite dimensional and F is algebraically closed, prove that every eigenvalue of $S \otimes T$ it the product of an eigenvalue of S with an eigenvalue of T.
   iii) Show, by constructing a counterexample with finite-dimensional vector spaces S, T over $\mathbf{R}$, that (ii) no longer holds when the hypothesis on F is dropped.

5. Let V be a finite-dimensional vector space over an algebraically closed field F, and fix $A, B \in \mathcal{L}(V)$. Consider the linear operator $T = T_{A,B} : X \mapsto AX + XB$ on $\mathcal{L}(V)$.

i) Express T in terms of tensor products (via the identification of $\mathcal{L}(V)$ with $V^* \otimes V$).

ii) Describe the eigenvalues of T in terms of the eigenvalues of A and B.

iii) Prove that if $F = \mathbf{C}$ and all eigenvalues of A, B has positive real part then every $M \in \mathcal{L}(V)$ can be written uniquely as $AX + XB$ for some $X \in \mathcal{L}(V)$.

Apropos eigenstuff... The next result generalizes what we proved in class about involutions (which are the special case $m = 2$, $\lambda_i = \pm 1$).

6. Suppose V is a vector space over a field F and T is a linear operator on V such that $\prod_{i=1}^{m}(T - \lambda_i I) = 0$ for some *distinct* $\lambda_i \in F$. Prove that V is the direct sum of the λ_i-eigenspaces of T. [NB: V may not be assumed finite-dimensional.]

Tensor products of A-modules. Like direct sums, quotient spaces, and duals, tensor products can be defined in the same way for modules over rings A that need not be fields. Basic properties such as $M \otimes (N \oplus N') \cong (M \otimes N) \oplus (M \otimes N')$ hold in this more general setting, and for much the same reason; but some new phenomena emerge, as in parts (ii) and (iii) of the next problem:

7. i) Show that if A is a commutative ring with unit, and $I \subseteq A$ is an ideal (an additive subgroup such that $aI \subseteq I$ for all $a \in A$, or equivalently a submodule of the A-module A), then $(A/I) \otimes_A (A/I)$, the tensor product of the quotient A-module A/I with itself, is isomorphic with A/I.

ii) On the other hand, show that $(\mathbf{Z}/2\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{Z}/3\mathbf{Z})$ is the trivial $\mathbf{Z}$-module $\{0\}$.

iii) For positive integers m, n, what is the $\mathbf{Z}$-module $(\mathbf{Z}/m\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{Z}/n\mathbf{Z})$?

Finally, a bit about inner products:

8. Solve Exercises 7 and 13 on pages 122, 123 of Axler. For #13, V is either a real or complex inner-product space, which need not be finite dimensional.

9. Is the symmetric bilinear pairing constructed in Problem 2 nondegenerate? When $F = \mathbf{R}$, is it positive definite?

Axler's exercise #7, as well as the more familiar #6, is often referred to as the "polarization identity". This shows that a linear transformation preserves the norm if and only if it preserves the inner product [more precisely, it shows the harder, "only if" part of this result]. These are basically also the identities used to prove Propositions 2 and 4 in the next chapter (pages 129, 130).

This problem set is due Friday, 8 October, at the beginning of class.