Math 55a: Honors Abstract Algebra

Homework Assignment #3 (17 September 2010):
Linear Algebra III

> The expression δ_{ij} [see problem 8] is called the Kronecker delta (after the mathematician Leopold Kronecker [1823–1891], who made more substantial contributions to mathematics than this).
> — Corwin and Szczarba, *Calculus in Vector Spaces*, p.124

A bit more about the structure of infinite-dimensional vector spaces:

1. i) Prove that a vector space with a countable[1] spanning set over a countable field is countable.
   ii) Prove that a vector space with a countable spanning set over any field does not have an uncountable linearly independent set.

2. i) Prove that if F is countable then F^∞ does not have a countable spanning set.
   ii) Prove that if F is uncountable then F^∞ does not have a countable spanning set.

Problem 2ii may be tricky. Can you give a proof that F^∞ is not countably generated that does not depend on the cardinality of F?

Some basics about linear transformations and their matrices:

3.–4. Solve Exercises 6, 22, 23, 24 from Chapter 3 of the textbook (pages 59 and 61). For #6, if $S_1 \cdots S_n$ is injective, what if anything can be said of $S_1, S_2, \ldots, S_n$? For the other three exercises, note that "$\mathcal{L}(V)$" is Axler's abbreviation for "$\mathcal{L}(V,V)$" (it is also known as $\mathrm{End}(V) = \mathrm{Hom}(V,V)$).

5. Let $\mathcal{P}_n$ be the ($\mathbf{R}$- or $\mathbf{C}$-)vector space of polynomials of degree at most n, and $L : \mathcal{P}_n \to \mathcal{P}_n$ be the linear transformation taking any polynomial $P(x)$ to the polynomial
$$(L(P))(x) = (x-3)P''(x)$$
(here P'' is the second derivative d^2P/dx^2). Exhibit a matrix for L relative to a suitable basis for $\mathcal{P}_n$, and determine the kernel, image, and rank of L.

6. Let V, W be arbitrary vector spaces over the same field. Show that, for any vector v in V, the evaluation map $E_v : \mathcal{L}(V,W) \to W$ defined by $E_v(L) = L(v)$ for all $L \in \mathcal{L}(V,W)$ is a linear transformation. If V, W are finite dimensional, what is the dimension of $\ker E_v$?

7. Let V, W be vector spaces over the rational field $\mathbf{Q}$. Prove that a map $T : V \to W$ is linear if and only if $T(v+v') = Tv + Tv'$ for all $v, v' \in V$. (Cf. the marginal note to Exercise 2 on p.59 of the textbook.)

[1] For us "countable" means "finite or countably infinite".

More about duality:

8. If $v_1, \ldots, v_n$ is a basis for V, prove that there is for each $j = 1, \ldots, n$ a unique $v_j^* \in V^*$ such that $v_j^*(v_i)$ is 1 if $i = j$ and 0 otherwise. [In other words, $v_j^*(v_i) = \delta_{ij}$, the "Kronecker delta" referred to in the Szczarba quote, which is also the (i, j) entry of the identity matrix.] Show further that the v_j^* constitute a basis for V^*. This is called the *dual basis* to $(v_1, \ldots, v_n)$.

9. We saw that, for any vector spaces V, W, the dual of $V \oplus W$ is naturally identified with $V^* \oplus W^*$. What is the dual of $\oplus_{i \in I} V_i$? Use this to construct a vector space V over some field F such that V is <u>not</u> isomorphic with V^*.

10. Let $x_0, \ldots, x_m$ be distinct elements of F. Recall that the $m+1$ vectors $v_i := (x_0^i, x_1^i, \ldots, x_m^i)$ $(0 \le i \le m)$ constitute a basis of F^{m+1}. Describe the dual basis.

Problem set is due Friday, Sep. 24 in class.