Math 55a: Honors Abstract Algebra

Homework Assignment #2 (10 September 2010):
Linear Algebra II

> **TFAE (The Following Are Equivalent)**: If I say this it means that, and if I say that it means the other thing, and if I say the other thing. . . [1]

This homework assignment consists of three problems on the basic definitions and properties of vector spaces and their dimensions, four problems showing these ideas in different contexts, and a bit more about modules. Problem set is due Friday, Sep. 17 in class.

1. Solve Exercise 7 on page 35 of the Axler textbook for vector spaces over an arbitrary field $\mathbf{F}$; deduce Exercises 5 and 6. ("$\mathbf{F}^\infty$" is the vector space defined on page 10; NB this is $\prod_{j=1}^{\infty} \mathbf{F}$, not $\oplus_{j=1}^{\infty}\mathbf{F}$.)

2. What is the dimension of the vector space generated by. . .
i) The vectors $(1, i, 1 + i, 0)$, $(-i, 1, 1 - i, 0)$, and $(1 - i, 1 + i, 2, 0)$ in $\mathbf{C}^4$?
ii) The functions $f(x) = \sin(x)$, $\sin(x + \pi/6)$, and $\sin(x + \pi/3)$ in $\mathbf{R}^{\mathbf{R}}$?
iii) The n vectors
$$\begin{array}{l}(1, 1, 0, 0, 0, \ldots, 0, 0, 0),\\ (0, 1, 1, 0, 0, \ldots, 0, 0, 0),\\ (0, 0, 1, 1, 0, \ldots, 0, 0, 0),\\ \qquad\qquad\vdots\\ (0, 0, 0, 0, 0, \ldots, 1, 1, 0),\\ (0, 0, 0, 0, 0, \ldots, 0, 1, 1),\\ (1, 0, 0, 0, 0, \ldots, 0, 0, 1)\end{array}$$
in $\mathbf{R}^n$ $(n \geq 2)$? [Note the first coordinate of the last vector!]

3. Find a basis for:
i) The subspace $\{\vec{x} : x_1 + \cdots + x_n = 0\}$ of F^n (this is the space we called "F_0^n").
ii) The subspace $\{P : P(-3) = 0\}$ of $\mathcal{P}$.
(In each case there are many right answers — but even more wrong ones.)

4. (Polynomial interpolation.) Let $a_1, a_2, \ldots, a_n$ be distinct elements of a field F. Prove that the following are equivalent:
i) For any $p_1, p_2, \ldots, p_n \in F$ there exists a unique polynomial $P(x)$ of degree less than n such that $P(a_i) = p_i$ for each $i = 1, 2, \ldots, n$.
ii) The n vectors $v_i := (a_1^i, a_2^i, \ldots, a_n^i)$ $(0 \leq i < n)$ in F^n are linearly independent.
Then prove one (and thus both) of those statements. [*Do not use determinants* even if you have seen them already!]

5. (Complexification of a real vector space.) Prove that for any $\mathbf{R}$-vector space V one may regard $V \oplus V$ as a $\mathbf{C}$-vector space $V_{\mathbf{C}}$ by defining the scalar multiplication by
$$(a + ib)(v, w) = (av - bw, aw + bv)$$

[1] *Definitions of Terms Commonly Used in Higher Math*, R. Glover et al.

for all $a, b \in \mathbf{R}$ and $v, w \in V$. [We shall later see that this is an example of a "tensor product", namely $V_{\mathbf{C}} = V \otimes_{\mathbf{R}} \mathbf{C}$.] If V is finite dimensional, what is the dimension of $V_{\mathbf{C}}$ as a $\mathbf{C}$-vector space?

6. Let V be a vector space over a field F, and let K a subfield of F (i.e., a subset containing $0, 1$ which is closed under $\pm$, $\times$, and multiplicative inverse, and thus constitutes a field with the arithmetic operations defined by restriction from K). Note that V and F may also be regarded as K-vector spaces by restricting the arithmetic operations appropriately. (For instance, $\mathbf{C}$ is an $\mathbf{R}$-vector space of dimension 2, and any $\mathbf{C}$-vector space is automatically an $\mathbf{R}$-vector space as well.) Show that if $m = \dim_K(F)$ and $n = \dim_F(V)$ are finite, then so is $d = \dim_K(V)$, and express d in terms of m and n.

7. Let F be a finite field, and let q be the number of elements of F. For some positive integer n consider the F-vector space $V = F^n$.
   i) Prove that V has q^n elements and

$$\prod_{i=0}^{n-1}(q^n - q^i) = (q^n - 1)(q^n - q)(q^n - q^2)\cdots(q^n - q^{n-1})$$

   ordered bases $(e_1, e_2, \ldots, e_n)$.
   ii) Now let k be an integer such that $0 \le k \le n$. How many subspaces of dimension k does V have?

We noted that the basic property $av = 0 \Rightarrow a = 0$ or $v = \vec{0}$ of vector spaces fails for modules. But often in mathematics such "failures" are also opportunities ...
Suppose M is a module over $\mathbf{Z}$, which is to say an abelian group. For an integer $n \neq 0$, we say $x \in M$ is an <u>n-torsion element</u> if $nx = 0$, and a <u>torsion element</u> if it is n-torsion for some n. Let $M[n]$ be the set of n-torsion elements, and $M_{\text{tors}} = \cup_{n \neq 0} M[n]$ the set of torsion elements. These are called the "n-torsion subgroup" and "torsion subgroup" of M, a terminology justified by the first part of the next problem.

8. i) Prove that $M[n]$ is a subgroup of M for each n, and that M_{tors} is a subgroup of M.
   ii) How might (i) fail if we use the same definitions of $M[n]$ and M_{tors} for a module M over an arbitrary commutative ring A (with n now allowed to be any nonzero element of A)? How can you fix the definitions so (i) works (and still gives something new — a "torsion submodule") in this generality?

9. An abelian group M is said to be *divisible* if for every nonzero integer n the map $M \to M$, $x \mapsto nx$ is surjective (i.e. the equation $nx = b$ has a solution $x \in M$ for all $b \in M$).
   i) Give an example of a divisible group $M \neq \{0\}$ whose only torsion element is 0.
   ii) Give an example of a divisible group $M \neq \{0\}$ such that $M = M_{\text{tors}}$. (A group M satisfying $M = M_{\text{tors}}$ is said to be a "torsion group".)