Math 55a: Honors Advanced Calculus and Linear Algebra

Metric topology VI: Cauchy sequences, completeness, and a third formulation of compactness

Cauchy sequences and total boundedness. [See Rudin, p.52 ff.] We noted already that convergence of sequences is not an intrinsic notion: $\{2^{-n}\} = 1/2, 1/4, 1/8, \ldots$ converges as a sequence in $\mathbf{R}$ but not in $(0,1)$. A closely related notion which is intrinsic is that of a *Cauchy sequence*. A sequence $\{p_n\}$ in a metric space X is Cauchy if for every $\epsilon > 0$ there is an integer N such that $d(p_m, p_n) < \epsilon$ for all $m, n \geq N$. (This can be rephrased as "$d(p_m, p_n) \to 0$ as $m, n \to \infty$ independently.") This notion is intrinsic because it does not require a limit point p whose existence may depend on the choice of ambient space. However, *any convergent sequence is Cauchy* [Rudin, 3.11a, p.53]: if $p_n \to p$, and $d(p_n, p) < \frac{1}{2}\epsilon$ for all $n \geq N$, then $d(p_m, p_n) < \epsilon$ for all $m, n \geq N$ by the triangle inequality.

Recall that X is said to be *totally bounded* if it has an ϵ-net for each $\epsilon > 0$. We have:

Theorem. *X is totally bounded if and only if every sequence in X has a Cauchy subsequence.*

Proof: ($\Leftarrow$) [This argument should look familiar.] We may assume $X \neq \emptyset$. Given $\epsilon > 0$, we inductively construct an ϵ-net $\{p_1, \ldots, p_N\}$ as follows. Choose p_1 arbitrarily. Having chosen $p_1, \ldots, p_n$, if $\{p_1, \ldots, p_n\}$ is not yet an ϵ-net, let p_{n+1} be a point such that $d(p_m, p_{n+1}) \geq \epsilon$ for each $m = 1, \ldots, n$. We claim that this process must terminate. Indeed, if it didn't, we would obtain a sequence $\{p_n\}_{n=1}^{\infty}$ any two of whose points are at distance $\geq \epsilon$ from each other; but such a sequence clearly has no Cauchy subsequence.

($\Rightarrow$) Let S_k be a $(1/2k)$-net for $k = 1, 2, 3, \ldots$ Given a sequence $\{p_n\}$ in X, extract a subsequence $\{q_n^{(1)}\}$ contained in a radius-$(1/2)$ neighborhood about one of the points in S_1. From $\{q_n^{(1)}\}$ extract a subsequence $\{q_n^{(2)}\}$ contained in a radius-$(1/4)$ neighborhood about one of the points in S_2. Keep going inductively: $\{q_n^{(k)}\}$ is a subsequence of $\{q_n^{(k-1)}\}$ contained in a radius-$(1/2k)$ neighborhood about one of the points in S_k. This is possible because for each k the sequence $\{q_n^{(k-1)}\}$ is infinite while S_k is finite. Note that $d(q_n^{(k)}, q_{n'}^{(k)}) < 1/k$ for all k, n, n'. Now consider the *diagonal subsequence* $\{q_k^{(k)}\}$. This is a subsequence of the original $\{p_n\}$; moreover, if $m, n \geq N$ then $q_m^{(m)}$ and $q_n^{(n)}$ are both contained in the subsequence $\{q_n^{(N)}\}$, and thus are at distance $< 1/N$. Thus $\{q_k^{(k)}\}$ is a Cauchy subsequence of $\{p_n\}$. $\square$

Completeness and Compactness. A metric space X is said to be *complete* if every Cauchy sequence in X converges in X. If X, Y are nonempty metric spaces then $X \times Y$ is complete if and only if X and Y are complete. Similarly, if S is a nonempty set then $\mathcal{B}(S, Y)$ is complete if and only if Y is. If X is complete, then *a subset $E \subseteq X$ is complete if and only if it is closed.* Indeed we have seen that E is closed iff, for every sequence $\{p_n\}$ in E converging to some $p \in X$, its limit p is also contained in E. But by hypothesis $\{p_n\}$ converges in X if and only if it is Cauchy — and since that's an intrinsic notion, $\{p_n\}$ is Cauchy in X if and only if it is Cauchy in E. So, E is closed iff every Cauchy sequence in E converges in E, which is precisely the condition for E to be complete. As a corollary, if Y is complete and X is any topological space, $\mathcal{C}(X, Y)$ is complete, because we have already seen that it is closed as a subset of $\mathcal{B}(X, Y)$.

It is a fundamental fact that $\mathbf{R}$ *is complete* — from which it follows that $\mathbf{R}^k$ ($k = 1, 2, 3, \ldots$), and any closed subset of $\mathbf{R}^k$, is complete. Moreover, *any bounded subset of $\mathbf{R}^k$ is totally bounded*, because such a set is contained in some $N_M(x)$, and the set of $y \in N_M(x)$ such that each coordinate of $y - x$ is in $\epsilon\mathbf{Z}$ is an ϵ-net for each $\epsilon > 0$. [Note that we're implicitly using the result of Problem 7 in the third homework set here: if $E \subseteq X$ is totally bounded relative to X, then E is totally bounded.] This, together with our above characterization of totally bounded sets, yields the **Heine-Borel Theorem** [Rudin, p.39–40]: *A subset of $\mathbf{R}^k$ is compact if and only if it is closed and bounded.* More generally, we have our third equivalent definition of compactness, in the context of subsets of a *complete* metric space:

Theorem. *A subset of a complete metric space is compact if and only if it is closed and totally bounded.*

Proof: Given our work thus far we need only show that a sequentially compact metric space is complete. But if $\{p_n\}$ is a Cauchy sequence then the limit of any convergent subsequence is also the limit of the entire sequence. $\Box$

Applications to continuous functions. We noted already that the continuous image of a compact set is compact. In particular it is bounded and closed. So, for instance, if X is compact and Y is any metric space then $\mathcal{C}(X, Y)$ consists simply of all continuous functions $f : X \to Y$, since any such function is automatically bounded. In the important special case $Y = \mathbf{R}$, we have seen that bounded closed sets contain their sup and inf; so, for instance, *any continuous real-valued function on a nonempty compact space attains its supremum and infimum.* We shall use this repeatedly to prove many results from Rolle's theorem (which undergirds the entire theory of Taylor expansions) to the spectral theorem for Hermitian operators on finite-dimensional inner product spaces. Going in another direction, for any $f, g \in \mathcal{C}(X, Y)$ we may consider the continuous function $x \mapsto d(f(x), g(x))$ to show that when X is compact the distance in $\mathcal{C}(X, Y)$ is given by $d(f, g) = \max_{x \in X} d(f(x), g(x))$ [NB the usual sup, to substitute for max when X is infinite, is not needed in the compact case].

If X, Y are metric spaces with X compact then *any continuous $f : X \to Y$ is uniformly continuous.* Indeed, given $\epsilon > 0$ the preimages of $(\epsilon/2)$-neighborhoods in Y constitute an open cover of X, which has a Lebesgue number $\delta > 0$ by the 8th problem on the third homework set (the Lebesgue covering lemma). Thus each $f(N_\delta(x))$ $(x \in X)$ is contained in some $N_{\epsilon/2}(y)$, so $d(x, x') < \delta \Rightarrow d(f(x), f(x')) < \epsilon$ as claimed.

Completions. How did we know in the first place that $\mathbf{R}$ is complete? The answer to that depends on how you define the real numbers — various approaches make this result a theorem, an axiom, or even a definition. [For further details, see Rudin, Chapter 1, and further references such as given in Rudin, p.21.] If you believe in sup and inf, you can obtain the limit of a real Cauchy sequence $\{p_n\}$ as either $\limsup p_n$ or $\liminf p_n$, these being defined for any bounded sequence of $s_n \in \mathbf{R}$ by

$$\limsup p_n := \inf_{n=1}^{\infty}\left(\sup_{m\geq n} s_m\right), \quad \liminf p_n := \sup_{n=1}^{\infty}\left(\inf_{m\geq n} s_m\right).$$

(Note that the sequences $\{\sup_{m\geq n} s_m\}_{n=1}^{\infty}$ and $\{\inf_{m\geq n} s_m\}_{n=1}^{\infty}$ are monotonically decreasing and increasing respectively — see [Rudin, p.55] for the definition if necessary.)

One nice approach is to obtain $\mathbf{R}$ as a "completion" of $\mathbf{Q}$. This approach is implicit in writing an arbitrary real number as a possibly nonterminating decimal, with an ambiguity in cases such as $54.9999\ldots = 55.0000\ldots$. Once this is done, it can be generalized by the following important construction. [See Rudin, Exercise 24 on p.82.] To any metric space X we associate its *completion*, which is a complete metric space X^* in which X is embedded isometrically as a dense subset. We construct X^* out of Cauchy sequences in X. For any two such sequences $\{p_n\}$, $\{q_n\}$, the sequence $\{d_n\} := \{d(p_n, q_n)\}$ in $\mathbf{R}$ is Cauchy, and thus has a limit, which we call the "distance" between $\{p_n\}$ and $\{q_n\}$. This "distance" satisfies all the axioms of a distance function, except that $d(\{p_n\}, \{q_n\}) = 0$ need not imply $\{p_n\} = \{q_n\}$. But, as a special case of Problem 5 of the first homework set, we get a genuine metric space by identifying any Cauchy sequences at distance 0 from each other. The resulting space is the completion X^*; it contains an isometric copy of X consisting of the equivalence classes of constant sequences $x, x, x, \ldots$. It is easy to see that this copy of X is dense in X^*. The only tricky thing to prove is that X^* is in fact complete: we must show that X^* contains limits of Cauchy sequences not only in X, but also in X^*. Suppose $\{x_n^*\}$ is such a Cauchy sequence. We find a Cauchy sequence in X whose limit in X^* is also the limit of $\{x_n^*\}$. For $k = 1, 2, 3, \ldots$ choose N_k so that $d(x_m^*, x_n^*) < 1/k$ for all $m, n \geq N_k$. Let $x_k \in X$ be chosen so $d(x_k, x_{N_k}^*) < 1/k$. One readily checks that $\{x_k\}$ is a Cauchy sequence at distance zero from $\{x_n^*\}$.

[This proof may also be viewed as a disguised "diagonal argument". Say that $\{x_i\}$ is a "fast Cauchy sequence" if $d(x_m, x_n) < 1/N$ whenever $m, n > N$. Clearly every Cauchy sequence has a fast subsequence. In particularly, each $x^* \in X^*$ is represented by a fast Cauchy sequence in X. Now suppose $\{x_n^*\}$ is a Cauchy sequence in X^*. For each n, choose a representative fast Cauchy sequence $\{x_{n,i}\}_{i=1}^{\infty}$. We claim: the diagonal sequence $\{x_{n,n}\}_{n=1}^{\infty}$ is a (not necessarily fast) Cauchy sequence in X whose limit is also the limit of $\{x_n^*\}$. To show that it's Cauchy we argue in much the same way that we proved the continuity of a uniform limit of continuous functions. For large enough N, we have

$$d(x_{m,m}, x_{n,n}) \leq \frac{1}{m} + \frac{1}{n} + d(x_m^*, x_n^*) < \frac{2}{N} + d(x_m^*, x_n^*)$$

for all $m, n > N$. Now choose N large enough that $1/N < \epsilon/3$ and that $d(x_m^*, x_n^*) < \epsilon/3$ for all $m, n > N$. Then $d(x_{m,m}, x_{n,n}) < \epsilon$. Now the distance between the Cauchy sequences $\{x_{n,n}\}$ and $\{x_n^*\}$ vanishes[1] because $d(x_{n,n}, x_n^*) \leq 1/n$. Hence they have the same limit. We have thus exhibited a limit of an arbitrary Cauchy sequence in X^*, completing the proof that X^* is complete.]

For instance, $\mathbf{R}$ is the completion of $\mathbf{Q}$, and $[0,1]$ is the completion of $(0,1)$. More generally, if X is already complete and E is any subset of X then the completion of E is naturally identified with the closure of E in X.

Example: N-adic numbers. There are other nice spaces that can be obtained by completing $\mathbf{Q}$ relative to a more exotic metric. For instance, suppose we say that two rational numbers r, r' are close to each other if their decimal expansions agree to many places to the *left* of the decimal point; more precisely, we define the "10-adic metric" d_{10} on $\mathbf{Q}$ as follows: if $r \neq r'$ we declare $d_{10}(r, r') = 10^{-e}$ where $r - r' = 10^e m/n$ for some integers m, n with n relatively prime to m as well as to 10. Of course $d_{10}(r, r) = 0$ for all r. One easily verifies that this is in fact a metric on $\mathbf{Q}$, and indeed satisfies not only the triangle inequality but the stronger "nonarchimedean triangle inequality" $d(p, q) \leq \max(d(p, r), d(q, r))$. For instance, with this metric Euler's nonsensical series $1! - 2! + 3! - 4! + - \cdots$ actually converges (while the usual series $\sum_{n=1}^{\infty} 1/n!$ for e badly diverges). Completing $\mathbf{Q}$ relative to d_{10} yields the "10-adic numbers" $\mathbf{Q}_{10}$. Of course there is nothing special about 10 here: one can likewise define N-adic numbers for any fixed integer $N > 1$. In fact the nicest case is d_N for N prime: one can add, subtract, and divide in $\mathbf{Q}_N$ for all N, just as one does in $\mathbf{R}$, but one can only divide by arbitrary nonzero numbers if N is prime — do you see why? The N-adic numbers may appear to be a pathological curiosity (e.g. for each N they have the topology of a countable union of disjoint Cantor sets), and indeed they were originally constructed as an amusement; but N-adic numbers and generalizations thereof are now ubiquitous tools in number theory and algebraic geometry.

[1] In mathematical writing "x vanishes" is a synonym for "$x = 0$".