**Math 55a: Honors Advanced Calculus and Linear Algebra**

Metric topology II: open and closed sets, etc.

**Neighborhoods (a.k.a. open balls) and open sets.** To further study and make use of metric spaces we need several important classes of subsets of such spaces. They can all be based on the notion of the *$r$-neighborhood*, defined as follows. Let $X$ be a metric space, $p \in X$, and $r > 0$. The $r$-neighborhood of $p$ is the set of all $q \in X$ at distance $< r$ from $p$:

$$N_r(p) := \{q \in X : d(p,q) < r\}$$

[Rudin, 2.18a, p.32]. Since Rudin's text was written, the equivalent term *open ball of radius $r$ about $p$* has come into more general use, and $N_r(p)$ is often called $B_r(p)$. This term is motivated by the shape of $N_r(p)$ when $X$ is $\mathbf{R}^3$ with the Euclidean metric. Here are some examples of $N_r(p)$ in other metric spaces: in $\mathbf{R}$, it is the "open interval" $(r-p, r+p)$; likewise in $\mathbf{R}^n$ with the sup metric, $N_r(p)$ is an open (hyper)cube of side $2r$ centered at $p$; if $d(\cdot,\cdot)$ is the discrete metric (see #1 on the first problem set), $N_r(p) = \{p\}$ or $X$ according as $r \leq 1$ or $r > 1$. Visualizing $N_r(p)$ for various $p \in X$ and $r > 0$ is a good way to get a feel for the metric space $X$.

Now let $E$ be any subset of $X$. The *interior points* of $E$ are those $p \in X$ some neighborhood of which is contained in $E$, i.e. those $p \in X$ for which there exists $r > 0$ such that $N_r(p) \subseteq E$ [Rudin, 2.18e]. Necessarily $p \in E$ (why?). The subset $E \subseteq X$ is said to be *open* in $X$ if and only if every point of $E$ is an interior point of $E$ [Rudin, 2.18f]. Note that, unlike the notion of boundedness, openness of $E$ depends not only on $E$ but also on the "ambient space" $X$. For instance, every metric space is open as a subset of itself, but a one-point subset of $\mathbf{R}$ cannot be open as a subset of $\mathbf{R}$ (check these assertions!). We shall only say/write statements like "$E$ is open" when the ambient space is clear from context.

Calling $N_r(p)$ an "open ball" would be horribly confusing if such sets $N_r(p)$ could fail to be open. The name is justified by the following result (Rudin, Thm. 2.19, p.32):

**Theorem.** *Every neighborhood is an open set.*

That is, for any metric space $X$, any $p \in X$, and any $r > 0$, the set $N_r(p)$ is open as a subset of $X$.

*Proof*: We must show that for any $q \in N_r(p)$ there is an $h > 0$ such that $N_h(q) \subseteq N_r(p)$. We claim that $h = r - d(p,q)$ works. Indeed, $h$ is positive by the definition of $N_r(p)$; and for any $s \in N_h(q)$ we have $s \in N_r(p)$ because

$$d(p,s) \leq d(p,q) + d(q,s) < (r-h) + h = r,$$

so $N_h(q)$ is a subset of $N_r(p)$ as desired. $\square$

A key fact about open sets is that a finite intersection of open sets is again open, as is an *arbitrary* union of open sets:

**Theorem.** *i) if $G_\alpha$ is an open subset of $X$ for each $\alpha \in I$, then so is $\cup_{\alpha \in I} G_\alpha$.*
*ii) If each of $G_1, \ldots, G_n$ is an open subset of $X$, then so is $\cap_{i=1}^n G_i$.*

[Rudin, Thm. 2.24 (p.34), a and c. In part (i), $I$ is an "index set" of arbitrary size. In part (ii), it is essential that the intersection be finite; a counterexample with a countably infinite intersection is $X = \mathbf{R}$, $G_n = N_{1/n}(0) = (-1/n, 1/n)$ $(n = 1, 2, 3, \ldots)$, when $\cap_{i=1}^\infty G_n = \{0\}$ is not open.]

*Proof*: (i) Put $G = \cup_{\alpha \in I} G_i$. To show $G$ is open, we must construct for each $x \in G$ a positive $r$ such that $N_r(x) \subseteq G$. Since $x \in G_\alpha$ for some $\alpha \in I$, we already have $r > 0$ such that $N_r(x) \subseteq G_\alpha$. Since $G \supseteq G_\alpha$, it follows that $N_r(x) \subseteq G_\alpha$ as was needed.
(ii) Put $H = \cap_{i=1}^n G_i$. To show $H$ is open, we must construct for each $x \in H$ a positive $r$ such that $N_r(x) \subseteq H$, i.e. such that $N_r(x) \subset G_i$ for each $i = 1, \ldots, n$. But each $G_i$ is open, so we have $r_1, \ldots, r_n$ such that $N_{r_i}(x) \subseteq G_i$ for each $i$. Let $r = \min(r_1, \ldots, r_n)$. Then $r > 0$ and $r \leq r_i$ for each $i$. Thus $N_r(x) \subseteq N_{r_i}(x)$, so $N_r(x) \subseteq G_i$, and we are done. $\square$

[For many purposes all that we'll need to know about the family $\mathcal{T}$ of open sets in $X$ is that $\mathcal{T}$ contains $\emptyset$ and $X$, the intersection of any $G_1, \ldots, G_n \in \mathcal{T}$, and an arbitrary union of $G_\alpha \in \mathcal{T}$. A family of subsets of a set $X$ which satisfies these three conditions, whether or not it arises as the open sets of some metric space, is called a *topology* on $X$, which then becomes a *topological space* $(X, \mathcal{T})$. Any result involving metric spaces which can be rephrased in terms of open sets and proved using only the above axioms on $\mathcal{T}$ is then valid in the larger category of topological spaces.]

**Closed sets and limit points.** A *closed* subset of a metric space $X$ is by definition the complement of an open subset. Using de Morgan's laws (the complement of an intersection is the union of the complements, and vice versa; see "Thm. 2.22" in Rudin, p.33–34) we immediately obtain:

**Theorem.** [Rudin, 2.24b,d]
*i) if $G_\alpha$ is a closed subset of $X$ for each $\alpha \in I$, then so is $\cap_{\alpha \in I} G_\alpha$.*
*ii) If each of $G_1, \ldots, G_n$ is a closed subset of $X$, then so is $\cup_{i=1}^n G_i$.*

Unwinding the definition, we see that $E \subseteq X$ is closed if and only if for every $p \notin E$ there exists $r > 0$ such that $N_r(p)$ is disjoint from $E$. The prototypical example of a closed set in $X$ is the *closed ball* of radius $r \geq 0$ about a point $p \in X$, defined by

$$\overline{B}_r(p) := \{q \in X : d(p, q) \leq r\}$$

(As with the openness of $N_r(p)$, this requires proof, which you can easily supply.) Note that $r = 0$ is allowed, with $\overline{B}_0(p)$ being simply $\{p\}$. In $\mathbf{R}$, the closed $r$-ball about $p$ is the "closed interval" $[r - p, r + p]$. Further examples of closed sets are $\emptyset$ and $X$ itself, and the complement $(N_r(p))^c = \{q \in X : d(p, q) \geq r\}$ of a neighborhood.

NB "closed" does <u>not</u> mean "not open"! A subset of a metric space might be both open and closed (as we already saw for $\emptyset$ and $X$, and also in #1 on the first problem set); it can also fail to be either open or closed (as with a "half-open interval" $[a, b) \subset \mathbf{R}$, or more dramatically $\mathbf{Q} \subset \mathbf{R}$).

You may notice that Rudin defines closed sets differently (2.18d, p.32), but then proves that the two definitions are equivalent (2.23, p.34). Rudin's definition involves the notion of a *limit point*. A point $p \in X$ is said to be a limit point of the subset $E \subseteq X$ if every neighborhood of $p$ contains a point of $E$ other than $p$ itself; i.e. if for all $r > 0$ there exists $q \in E$ such that $0 < d(p, q) < r$. Then

*$E$ is closed if and only if every limit point of $E$ is contained in $E$.*

*Proof*: Suppose $E$ is closed, and let $x$ be a limit point. We prove that $x \in E$ by contradiction. Assume that $x \notin E$. Since $x$ would then be in the complement of $E$, it would have a neighborhood $N_r(x)$ disjoint from $E$, contradicting the definition of a limit point. Therefore $x \in E$. We have thus shown that a closed set contains all its limit points.

Conversely, suppose $E$ contains all its limit points. Then any $x \notin E$ is not a limit point of $E$. Thus there exists $r > 0$ such that $N_r(x)$ contains no point of $E$. Therefore $E$ is closed. $\square$

An equivalent description of limit points is the following result (essentially a restatement of Rudin's "Theorem 2.20" on p.32–33):

**Theorem.** *$p$ is a limit point of $E$ if and only if there exist points $q_n \in E$ $(n = 1, 2, 3, \ldots)$, with each $q_n \neq p$, such that for every $r > 0$ we have $d(p, q_n) < r$ for all but finitely many $n$.*

*Proof*: ($\Leftarrow$) is clear, since "all but finitely many" certainly forces "at least one". For ($\Rightarrow$) we construct $q_n$ as follows: let $r = 1/n$ in the definition of limit point, and let $q_n$ be a point such that $0 < d(p, q_n) < 1/n$. Then for each $r > 0$ we have $r > 1/N$ for some integer $N$; then $d(p, q_n) < r$ once $n > N$, and there are only finitely many integers $n$ which do not exceed $N$. $\square$

[We shall see that the $q_n$ then constitute a *sequence* of points in $E \backslash \{p\}$ whose *limit* is $p$, once we define "sequence" and "limit" a few lectures hence.]

We also find [Rudin, p.33]:

**Theorem.** *A finite set has no limit points.*

Indeed, if $E$ is finite then for each $p \in X$ there are only finitely many $q \neq p$ in $E$, and thus finitely many distances $d(p, q)$. Thus if $r$ is smaller than the least of them then there is no $q \in E$ such that $0 < d(p, q) < r$. $\square$

**Closures.** For any subset $E$ of a metric space $X$, we define the *closure* $\overline{E}$ of $E$ to be the set of all $p \in X$ such that $p \in E$ or $p$ is a limit point of $E$ (or both). That is, $\overline{E} := E \cup E'$ where $E'$ is the set of all limits points of $E$ in $X$. Clearly

if $F \supseteq E$ then $F' \supseteq E'$ and thus $\overline{F} \supseteq \overline{E}$.

**Theorem.** [Rudin, 2.27, p.35] *For any subset $E$ of a metric space $X$,*
*i) $\overline{E}$ is closed.*
*ii) $E = \overline{E}$ if and only if $E$ is closed.*
*iii) $\overline{E} \subseteq F$ for every closed set $F \subseteq X$ such that $F \supseteq E$.*

[by (a) and (c), $\overline{E}$ is the *smallest* closed subset of $X$ that contains $E$, and the intersection of all closed $F \supseteq E$. NB this is a topological notion.]

*Proof*: (i) We must construct, for each $p \in X$ with $p \notin \overline{E}$, a neighborhood of $p$ disjoint from $\overline{E}$. Since $p$ is not a limit point of $E$, there exists $r > 0$ such that $E$ contains no point $q$ with $d(p, q) < r$ — note that we need not impose the usual constraint $q \neq p$, because we already assumed $p \notin \overline{E}$, and $\overline{E} \supseteq E$. Thus $N_r(p)$ is disjoint from $E$. We claim that it is also disjoint from $E'$. Indeed, suppose $q \in N_r(p)$. Since $N_r(p)$ is open, there exists $h > 0$ such that $N_h(q) \subseteq N_r(p)$. Thus $N_h(q)$ is disjoint from $E$, and $q$ is not a limit point of $E$, as claimed. We conclude that $N_r(p)$ is disjoint from $E \cup E' = \overline{E}$, as desired.
(ii) ($\Rightarrow$) if $E = \overline{E}$ then $E$ is closed by (i).
($\Leftarrow$) If $E$ is closed then we have seen $E' \subseteq E$, so $\overline{E} = E \cup E' = E$, as claimed.
(iii) We saw that if $F \supseteq E$ then $\overline{F} \supseteq \overline{E}$. But if $F$ is closed then $\overline{F} = F$ by (ii). Thus $F \supseteq \overline{E}$. $\square$

In particular $\overline{B}_r(p) \supseteq \overline{B_r(p)}$ for all $r > 0$ (since $\overline{B}_r(p)$ is an example of a closed set that contains $B_r(p)$). In $\mathbf{R}^n$ it is always true that $\overline{B}_r(p) = \overline{B_r(p)}$, but in some metric spaces $\overline{B}_r(p)$ may be strictly larger than $\overline{B_r(p)}$ for some $p, r$; do you see how this can happen?

Going back to $X = \mathbf{R}$, we have:

**Theorem.** [Rudin, 2.28, p.35] *Let $E \subset \mathbf{R}$ be a nonempty set bounded above. Then $(\sup E) \in \overline{E}$. In particular if $E$ is closed than $E \ni (\sup E)$.*

*Proof*: Let $y = \sup E$. We prove that $y \in \overline{E}$ by contradiction. Assume that $y \notin \overline{E}$. Since $\overline{E}$ is closed, there would then exist $h > 0$ such that $N_h(y)$ is disjoint from $\overline{E}$, and thus *a fortiori* from $E$. But then $y - h$ would be an upper bound on $E$ strictly smaller than $y$. This is a contradiction, and we conclude that $u \in \overline{E}$. $\square$

This result will be fundamental to our rigorous development of the differential calculus.