Math 55a: Honors Advanced Calculus and Linear Algebra

Metric topology II: open and closed sets, etc.

Neighborhoods (a.k.a. open balls) and open sets. To further study and make use of metric spaces we need several important classes of subsets of such spaces. They can all be based on the notion of the *r-neighborhood*, defined as follows. Let X be a metric space, $p \in X$, and $r > 0$. The r-neighborhood of p is the set of all $q \in X$ at distance $< r$ from p:

$$N_r(p) := \{q \in X : d(p, q) < r\}$$

[Rudin, 2.18a, p.32]. Since Rudin's text was written, the equivalent term *open ball of radius r about p* has come into more general use, and $N_r(p)$ is often called $B_r(p)$. This term is motivated by the shape of $N_r(p)$ when X is $\mathbf{R}^3$ with the Euclidean metric. Here are some examples of $N_r(p)$ in other metric spaces: in $\mathbf{R}$, it is the "open interval" $(r - p, r + p)$; likewise in $\mathbf{R}^n$ with the sup metric, $N_r(p)$ is an open (hyper)cube of side $2r$ centered at p; if $d(\cdot, \cdot)$ is the discrete metric (see #1 on the first problem set), $N_r(p) = \{p\}$ or X according as $r \leq 1$ or $r > 1$. Visualizing $N_r(p)$ for various $p \in X$ and $r > 0$ is a good way to get a feel for the metric space X.

Now let E be any subset of X. The *interior points* of E are those $p \in X$ some neighborhood of which is contained in E, i.e. those $p \in X$ for which there exists $r > 0$ such that $N_r(p) \subseteq E$ [Rudin, 2.18e]. Necessarily $p \in E$ (why?). The subset $E \subseteq X$ is said to be *open* in X if and only if every point of E is an interior point of E [Rudin, 2.18f]. Note that, unlike the notion of boundedness, openness of E depends not only on E but also on the "ambient space" X. For instance, every metric space is open as a subset of itself, but a one-point subset of $\mathbf{R}$ cannot be open as a subset of $\mathbf{R}$ (check these assertions!). We shall only say/write statements like "E is open" when the ambient space is clear from context.

Calling $N_r(p)$ an "open ball" would be horribly confusing if such sets $N_r(p)$ could fail to be open. The name is justified by the following result (Rudin, Thm. 2.19, p.32):

Theorem. *Every neighborhood is an open set.*

That is, for any metric space X, any $p \in X$, and any $r > 0$, the set $N_r(p)$ is open as a subset of X.

Proof: We must show that for any $q \in N_r(p)$ there is an $h > 0$ such that $N_h(q) \subseteq N_r(p)$. We claim that $h = r - d(p, q)$ works. Indeed, h is positive by the definition of $N_r(p)$; and for any $s \in N_h(q)$ we have $s \in N_r(p)$ because

$$d(p, s) \leq d(p, q) + d(q, s) < (r - h) + h = r,$$

so $N_h(q)$ is a subset of $N_r(p)$ as desired. $\Box$

A key fact about open sets is that a finite intersection of open sets is again open, as is an *arbitrary* union of open sets:

Theorem. *i) if G_α is an open subset of X for each $\alpha \in I$, then so is $\cup_{\alpha\in I} G_\alpha$. ii) If each of $G_1, \ldots, G_n$ is an open subset of X, then so is $\cap_{i=1}^n G_i$.*

[Rudin, Thm. 2.24 (p.34), a and c. In part (i), I is an "index set" of arbitrary size. In part (ii), it is essential that the intersection be finite; a counterexample with a countably infinite intersection is $X = \mathbf{R}$, $G_n = N_{1/n}(0) = (-1/n, 1/n)$ $(n = 1, 2, 3, \ldots)$, when $\cap_{i=1}^\infty G_n = \{0\}$ is not open.]

Proof: (i) Put $G = \cup_{\alpha\in I} G_i$. To show G is open, we must construct for each $x \in G$ a positive r such that $N_r(x) \subseteq G$. Since $x \in G_\alpha$ for some $\alpha \in I$, we already have $r > 0$ such that $N_r(x) \subseteq G_\alpha$. Since $G \supseteq G_\alpha$, it follows that $N_r(x) \subseteq G_\alpha$ as was needed.
(ii) Put $H = \cap_{i=1}^n G_i$. To show H is open, we must construct for each $x \in H$ a positive r such that $N_r(x) \subseteq H$, i.e. such that $N_r(x) \subset G_i$ for each $i = 1, \ldots, n$. But each G_i is open, so we have $r_1, \ldots, r_n$ such that $N_{r_i}(x) \subseteq G_i$ for each i. Let $r = \min(r_1, \ldots, r_n)$. Then $r > 0$ and $r \leq r_i$ for each i. Thus $N_r(x) \subseteq N_{r_i}(x)$, so $N_r(x) \subseteq G_i$, and we are done. $\square$

[For many purposes all that we'll need to know about the family $\mathcal{T}$ of open sets in X is that $\mathcal{T}$ contains $\emptyset$ and X, the intersection of any $G_1, \ldots, G_n \in \mathcal{T}$, and an arbitrary union of $G_\alpha \in \mathcal{T}$. A family of subsets of a set X which satisfies these three conditions, whether or not it arises as the open sets of some metric space, is called a *topology* on X, which then becomes a *topological space* $(X, \mathcal{T})$. Any result involving metric spaces which can be rephrased in terms of open sets and proved using only the above axioms on $\mathcal{T}$ is then valid in the larger category of topological spaces.]

Closed sets and limit points. A *closed* subset of a metric space X is by definition the complement of an open subset. Using de Morgan's laws (the complement of an intersection is the union of the complements, and vice versa; see "Thm. 2.22" in Rudin, p.33–34) we immediately obtain:

Theorem. [Rudin, 2.24b,d]
i) if G_α is a closed subset of X for each $\alpha \in I$, then so is $\cap_{\alpha\in I} G_\alpha$.
ii) If each of $G_1, \ldots, G_n$ is a closed subset of X, then so is $\cup_{i=1}^n G_i$.

Unwinding the definition, we see that $E \subseteq X$ is closed if and only if for every $p \notin E$ there exists $r > 0$ such that $N_r(p)$ is disjoint from E. The prototypical example of a closed set in X is the *closed ball* of radius $r \geq 0$ about a point $p \in X$, defined by

$$\overline{B}_r(p) := \{q \in X : d(p, q) \leq r\}$$

(As with the openness of $N_r(p)$, this requires proof, which you can easily supply.) Note that $r = 0$ is allowed, with $\overline{B}_0(p)$ being simply $\{p\}$. In $\mathbf{R}$, the closed r-ball about p is the "closed interval" $[r - p, r + p]$. Further examples of closed sets are $\emptyset$ and X itself, and the complement $(N_r(p))^c = \{q \in X : d(p, q) \geq r\}$ of a neighborhood.

NB "closed" does <u>not</u> mean "not open"! A subset of a metric space might be both open and closed (as we already saw for $\emptyset$ and X, and also in #1 on the first problem set); it can also fail to be either open or closed (as with a "half-open interval" $[a, b) \subset \mathbf{R}$, or more dramatically $\mathbf{Q} \subset \mathbf{R}$).

You may notice that Rudin defines closed sets differently (2.18d, p.32), but then proves that the two definitions are equivalent (2.23, p.34). Rudin's definition involves the notion of a *limit point*. A point $p \in X$ is said to be a limit point of the subset $E \subseteq X$ if every neighborhood of p contains a point of E other than p itself; i.e. if for all $r > 0$ there exists $q \in E$ such that $0 < d(p, q) < r$. Then

E is closed if and only if every limit point of E is contained in E.

Proof: Suppose E is closed, and let x be a limit point. We prove that $x \in E$ by contradiction. Assume that $x \notin E$. Since x would then be in the complement of E, it would have a neighborhood $N_r(x)$ disjoint from E, contradicting the definition of a limit point. Therefore $x \in E$. We have thus shown that a closed set contains all its limit points.

Conversely, suppose E contains all its limit points. Then any $x \notin E$ is not a limit point of E. Thus there exists $r > 0$ such that $N_r(x)$ contains no point of E. Therefore E is closed. $\Box$

An equivalent description of limit points is the following result (essentially a restatement of Rudin's "Theorem 2.20" on p.32–33):

Theorem. *p is a limit point of E if and only if there exist points $q_n \in E$ $(n = 1, 2, 3, \ldots)$, with each $q_n \neq p$, such that for every $r > 0$ we have $d(p, q_n) < r$ for all but finitely many n.*

Proof: ($\Leftarrow$) is clear, since "all but finitely many" certainly forces "at least one". For ($\Rightarrow$) we construct q_n as follows: let $r = 1/n$ in the definition of limit point, and let q_n be a point such that $0 < d(p, q_n) < 1/n$. Then for each $r > 0$ we have $r > 1/N$ for some integer N; then $d(p, q_n) < r$ once $n > N$, and there are only finitely many integers n which do not exceed N. $\Box$

[We shall see that the q_n then constitute a *sequence* of points in $E \backslash \{p\}$ whose *limit* is p, once we define "sequence" and "limit" a few lectures hence.]

We also find [Rudin, p.33]:

Theorem. *A finite set has no limit points.*

Indeed, if E is finite then for each $p \in X$ there are only finitely many $q \neq p$ in E, and thus finitely many distances $d(p, q)$. Thus if r is smaller than the least of them then there is no $q \in E$ such that $0 < d(p, q) < r$. $\Box$

Closures. For any subset E of a metric space X, we define the *closure* $\overline{E}$ of E to be the set of all $p \in X$ such that $p \in E$ or p is a limit point of E (or both). That is, $\overline{E} := E \cup E'$ where E' is the set of all limits points of E in X. Clearly if $F \supseteq E$ then $F' \supseteq E'$ and thus $\overline{F} \supseteq \overline{E}$.

Theorem. [Rudin, 2.27, p.35] *For any subset E of a metric space X,*
i) $\overline{E}$ is closed.
ii) $E = \overline{E}$ if and only if E is closed.
iii) $\overline{E} \subseteq F$ for every closed set $F \subseteq X$ such that $F \supseteq E$.

[by (a) and (c), $\overline{E}$ is the *smallest* closed subset of X that contains E, and the intersection of all closed $F \supseteq E$. NB this is a topological notion.]

Proof: (i) We must construct, for each $p \in X$ with $p \notin \overline{E}$, a neighborhood of p disjoint from $\overline{E}$. Since p is not a limit point of E, there exists $r > 0$ such that E contains no point q with $d(p,q) < r$ — note that we need not impose the usual constraint $q \neq p$, because we already assumed $p \notin \overline{E}$, and $\overline{E} \supseteq E$. Thus $N_r(p)$ is disjoint from E. We claim that it is also disjoint from E'. Indeed, suppose $q \in N_r(p)$. Since $N_r(p)$ is open, there exists $h > 0$ such that $N_h(q) \subseteq N_r(p)$. Thus $N_h(q)$ is disjoint from E, and q is not a limit point of E, as claimed. We conclude that $N_r(p)$ is disjoint from $E \cup E' = \overline{E}$, as desired.
(ii) ($\Rightarrow$) if $E = \overline{E}$ then E is closed by (i).
($\Leftarrow$) If E is closed then we have seen $E' \subseteq E$, so $\overline{E} = E \cup E' = E$, as claimed.
(iii) We saw that if $F \supseteq E$ then $\overline{F} \supseteq \overline{E}$. But if F is closed then $\overline{F} = F$ by (ii). Thus $F \supseteq \overline{E}$. $\square$

In particular $\overline{B}_r(p) \supseteq \overline{B_r(p)}$ for all $r > 0$ (since $\overline{B}_r(p)$ is an example of a closed set that contains $B_r(p)$). In $\mathbf{R}^n$ it is always true that $\overline{B}_r(p) = \overline{B_r(p)}$, but in some metric spaces $\overline{B}_r(p)$ may be strictly larger than $\overline{B_r(p)}$ for some p, r; do you see how this can happen?

Going back to $X = \mathbf{R}$, we have:

Theorem. [Rudin, 2.28, p.35] *Let $E \subset \mathbf{R}$ be a nonempty set bounded above. Then $(\sup E) \in \overline{E}$. In particular if E is closed than $E \ni (\sup E)$.*

Proof: Let $y = \sup E$. We prove that $y \in \overline{E}$ by contradiction. Assume that $y \notin \overline{E}$. Since $\overline{E}$ is closed, there would then exist $h > 0$ such that $N_h(y)$ is disjoint from $\overline{E}$, and thus *a fortiori* from E. But then $y - h$ would be an upper bound on E strictly smaller than y. This is a contradiction, and we conclude that $u \in \overline{E}$. $\square$

This result will be fundamental to our rigorous development of the differential calculus.