Math 55a: Honors Advanced Calculus and Linear Algebra

Homework Assignment #8 (4 November 2005):
Linear Algebra IV — "Eigenstuff"
(and a projective overture)

HINT, *n.*: The hardest of several possible ways to do a proof.[1]

For a vector space V over a field F, the *projective space* $\mathbf{P}V$ is the set of 1-dimensional subspaces Of V. These are the "points" of $\mathbf{P}V$; its "lines" are the 2-dimensional subspaces, "planes" the 3-dimensional subspaces, and so on. In particular a "hyperplane" is a subspace of codimension 1. All of these may be regarded as projective spaces in their own right, containing some of the points, lines, etc. of $\mathbf{P}V$. In particular, if $\dim V = 3$ then $\mathbf{P}V$ is called a "projective plane", and likewise a "projective space of dimension n" is $\mathbf{P}V$ with $\dim(V) = n + 1$.

1. i) Show that for any two distinct points in a projective space there is a unique line containing them both, and that any two distinct lines in a projective *plane* meet in a unique point.

   ii) Let $\mathbf{P}V$ be a projective space of dimension n. Its *dual projective space* is $\mathbf{P}(V^*)$. The annihilator gives, for each $d = 0, 1, \ldots, n-1$, a natural bijection between the d-dimensional projective subspaces of $\mathbf{P}V$ and the $(n-1-d)$-dimensional projective subspaces of $\mathbf{P}(V^*)$. Show that this bijection is incidence-reversing: for subspaces U, U' of $\mathbf{P}V$, we have $U \subseteq U'$ if and only if $U^\perp \supseteq U'^\perp$.

   iii) A *polarity* of a finite-dimensional projective space $\mathbf{P}V$ is an isomorphism between $\mathbf{P}V$ and $\mathbf{P}(V^*)$ coming from a linear bijection $V \to V^*$. Composing a polarity with the construction of part (ii), we may associate with each "point" of $\mathbf{P}V$ its polar hyperplane. *If n is odd*, construct a polarity for which each point is in its own polar hyperplane. If $F = \mathbf{R}$, construct a polarity for which no point is in its own polar hyperplane.

We shall see that the parity condition in (iii) is necessary. The properties in (i) characterize the points and lines of a combinatorial projective plane. If F is finite, say $|F| = q$, then each line contains $q + 1$ points and each point is on $q + 1$ lines (why?), giving a "finite projective plane of order q". This construction works whenever q is a prime power. It is conjectured that if q is not a prime power then there is no

[1] *Definitions of Terms Commonly Used in Higher Math*, R. Glover et al.; cf. also Prob. 12.

finite projective plane of order q; but even the existence of a finite projective plane of order 12 is still an unsolved problem. It is known that for some prime powers q there are finite projective planes of order q that are not isomorphic with $\mathbf{P}V$.

2.–11. Solve Exercises 4, 7–12, 15, 16, 21 from Chapter 5 of the textbook (pages 94,95). As usual, $\mathbf{F}$ can be any field, and $\mathbf{C}$ can be any algebraically closed field. Do not assume that vector spaces are finite dimensional unless you must. For #4, remember that Axler's "null (T)" is our "ker (T)". For #16, how much of #15 remains true over an arbitrary field?

[#15 has the following important consequence: if $P \in F[z]$ and $P(T) = 0$ for some linear operator $T \in \mathrm{End}(V)$, then every eigenvalue of T is a root of P. For instance, the only possible eigenvalues of a linear involution are ± 1, the roots of $z^2 - 1$.]

12. Let A be the 2×2 matrix $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Compute the eigenvalues and eigenvectors of the linear transformation corresponding to A. Use these to find an invertible 2×2 matrix M such that MAM^{-1} is diagonal, and deduce a closed form for (the entries of) A^t as functions of $t \in \mathbf{Z}$. Check that your answers agree with the numbers you get by computing A^t directly for $|t| \leq 5$.

This problem set is due Monday [sic], 14 November, at the beginning of class.