Math 55a: Honors Advanced Calculus and Linear Algebra

Homework Assignment #6 (21 October 2005):
Linear Algebra II

TFAE (The Following Are Equivalent): If I say this it means that, and if I say that it means the other thing, and if I say the other thing...[1]

This homework assignment consists of three problems on the basic definitions and properties of vector spaces and their dimensions, and four problems showing these ideas in different contexts. Problem set is due Friday, Oct. 28 in class.

1. Solve Exercise 7 on page 35 of the Axler textbook for vector spaces over an arbitrary field $\mathbf{F}$; deduce Exercises 5 and 6. ("$\mathbf{F}^\infty$" is the vector space defined on page 10; NB this is $\prod_{j=1}^\infty \mathbf{F}$, not $\oplus_{j=1}^\infty \mathbf{F}$.)

2. What is the dimension of the vector space generated by...
   i) The vectors $(1, i, 1+i, 0)$, $(-i, 1, 1-i, 0)$, and $(1-i, 1+i, 2, 0)$ in $\mathbf{C}^4$?
   ii) The functions $f(x) = \sin(x)$, $\sin(x + \pi/6)$, and $\sin(x + \pi/3)$ in $\mathcal{C}(\mathbf{R})$?
   iii) The n vectors

$$\begin{array}{l}(1,1,0,0,0,\ldots,0,0,0),\\(0,1,1,0,0,\ldots,0,0,0),\\(0,0,1,1,0,\ldots,0,0,0),\\ \qquad\qquad\vdots\\(0,0,0,0,0,\ldots,1,1,0),\\(0,0,0,0,0,\ldots,0,1,1),\\(1,0,0,0,0,\ldots,0,0,1)\end{array}$$

   in $\mathbf{R}^n$ $(n \geq 2)$? [Note the first coordinate of the last vector!]

3. Find a basis for:
   i) The subspace $\{\vec{x} : x_1 + \cdots + x_n = 0\}$ of F^n (this is the space we called "F_0^n").
   ii) The subspace $\{P : P(-3) = 0\}$ of $\mathcal{P}$.
   (In each case there are many right answers — but even more wrong ones.)

[1] *Definitions of Terms Commonly Used in Higher Math*, R. Glover et al.

4. (Polynomial interpolation.) Let $a_1, a_2, \ldots a_n$ be distinct elements of a field F. Prove that the following are equivalent:
   i) For any $p_1, p_2, \ldots, p_n \in F$ there exists a unique polynomial $P(x)$ of degree less than n such that $P(a_i) = p_i$ for each $i = 1, 2, \ldots, n$.
   ii) The n vectors $v_i := (a_1^i, a_2^i, \ldots, a_n^i)$ $(0 \le i < n)$ in F^n are linearly independent.
   Then prove one (and thus both) of those statements. [*Do not use determinants* even if you have seen them already!]

5. (Complexification of a real vector space.) Prove that for any $\mathbf{R}$-vector space V one may regard $V \oplus V$ as a $\mathbf{C}$-vector space $V_{\mathbf{C}}$ by defining the scalar multiplication by
$$(a + ib)(v, w) = (av - bw, aw + bv)$$
   for all $a, b \in \mathbf{R}$ and $v, w \in V$. If V is finite dimensional, what is the dimension of $V_{\mathbf{C}}$ as a $\mathbf{C}$-vector space?

6. Let V be a vector space over a field F, and let K a subfield of F (i.e., a subset containing $0, 1$ which is closed under $\pm$, $\times$, and multiplicative inverse, and thus constitutes a field with the arithmetic operations defined by restriction from K). Note that V and F may also be regarded as K-vector spaces by restricting the arithmetic operations appropriately. (For instance, $\mathbf{C}$ is an $\mathbf{R}$-vector space of dimension 2, and any $\mathbf{C}$-vector space is automatically an $\mathbf{R}$-vector space as well.) Show that if $m = \dim_K(F)$ and $n = \dim_F(V)$ are finite, then so is $d = \dim_K(V)$, and express d in terms of m and n.

7. Let F be a finite field, and let q be the number of elements of F. For some positive integer n consider the F-vector space $V = F^n$.
   i) Prove that V has q^n elements and
$$\prod_{i=0}^{n-1} (q^n - q^i) = (q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{n-1})$$
   ordered bases $(e_1, e_2, \ldots, e_n)$.
   ii) Now let k be an integer such that $0 \le k \le n$. How many dimension-k subspaces does V have?