Math 55a: Honors Advanced Calculus and Linear Algebra

Homework Assignment #5 (14 October 2005):
Linear Algebra I: vector space basics

$$\ast$$

—Axler, page 1 [why?]

Some basic problems (mostly from Chapter 1 of the Axler textbook) on vector spaces and their subsets, intersections, and sums. This problem set is due, together with and any problems you postponed from the previous problem set, on Friday, Oct. 21, at the beginning of class.

1.–6.[1] Solve problems 4 through 14 on pages 19 and 20 of the Axler textbook. (In problem 4, and problems 8 through 13, V is a vector space over an arbitrary field $\mathbf{F}$.) Which if any of these basic results would fail if $\mathbf{F}$ were replaced by $\mathbf{Z}$?

7. The *symmetric difference* $A \bigtriangleup B$ of two subsets A, B of a set X is defined to be the subset of X consisting of those elements of X contained in exactly one of A and B.

i) Prove that the *power set* $2^X := \{A : A \subseteq X\}$ of X becomes a vector space over the two-element field $\mathbf{Z}/2\mathbf{Z}$ by taking its zero element to be the empty set and defining vector addition by $A + B = A \bigtriangleup B$. (This is a generalization of the case $X = \{1, 2, \ldots, n\}$, when 2^X is just $(\mathbf{Z}/2\mathbf{Z})^n$; in that case, you may be familiar with $\bigtriangleup$ in the guise of "(bitwise) exclusive or", as opposed to the "inclusive or" which corresponds to the union of sets.)

ii) Let X now be a topological space, and recall a subset $A \subseteq X$ is said to be *clopen* if it is simultaneously closed and open. Prove that the clopen sets in X constitute a subspace of the $(\mathbf{Z}/2\mathbf{Z})$-vector space 2^X defined in part (i).

[1] These 11 problems, plus the question about "vector spaces over $\mathbf{Z}$", are sufficiently small and straightforward compared to our usual fare that I'm counting each as the equivalent of only half of a problem.