Math 55a: Honors Advanced Calculus and Linear Algebra

Homework Assignment #4 (7 October 2005):
Completeness, and compactness: the grand finale

Q: What did the mathematician say as ϵ approached zero?
A: "There goes the neighborhood."
—*Hoary math joke*

This final topology homework contains several standard problems on completeness and compactness, and several which show these concepts in action in various contexts.

More about separation properties in metric spaces and general topological spaces:

1. Let X be a metric space and A, B disjoint closed subsets. Prove that there exist disjoint open sets U, V such that $U \supseteq A$ and $V \supseteq B$.

2. A topological space X is said to be *normal* if it is Hausdorff and has the property that for every disjoint closed subsets A, B there exist disjoint open sets U, V such that $U \supseteq A$ and $V \supseteq B$. Show that if A, B are disjoint closed subsets of a normal space X then there exists a continuous function $f : X \to \mathbf{R}$ such that $f(X) \subseteq [0, 1]$ and $f(x) = 0$ if $x \in A$ while $f(x) = 1$ if $x \in B$.
   [Hint: If $S \subset [0, 1]$ is a dense subset then a function $f : X \to [0, 1]$ is specified completely by the sets $f^{-1}([s, 1])$ for $s \in S$.]

To put #2 in context: the distance function on a metric space gives us a ready source of continuous functions from the space to $\mathbf{R}$; in particular, enough such functions to "separate points": if $x_0 \neq x_1$ then there's a continuous function taking x_0 to 0 and x_1 to 1. [For instance, the function $x \mapsto d(x, x_0)/d(x_1, x_0)$ does the trick.] According to Problem 4, $\mathcal{C}(X, \mathbf{R})$ separates points also under the hypothesis of normality, which is weaker than (well, at least as weak as) metrizability by #3.

Problem 2 should be quite challenging. By contrast, problems 3–6, concerning alternative formulations of compactness, should be fairly routine. Use these problems to also practice good solution writing. Especially with concepts such as continuity and compactness with several equivalent definitions, one can easily fall into the habit of plowing ahead with the first approach that comes to mind, which may produce a correct but rather unenlightening solution. As an extreme example, having learned that a product of two continuous functions is continuous, one could still demonstrate the continuity of a product of three continuous functions with an ϵ-δ proof; such a solution, if correct, will earn you full marks but little sympathy. A more elegant solution has the long-term advantages of deeper understanding as well as ease of review as final exams approach, and the short-term advantages of being less error prone and easier on Thanos to grade.

3. A family $\mathcal{F}$ of subsets of a set X is said to have the *finite intersection property* if $F_1 \cap \ldots \cap F_n \neq \emptyset$ for any $F_1, \ldots, F_n \in \mathcal{F}$ (i.e., finite intersections in $\mathcal{F}$

are nonempty). Prove that a topological space is compact if and only if $\cap_{F\in\mathcal{F}}F \neq \emptyset$ for every family $\mathcal{F}$ of closed subsets of X with the finite intersection property.

4. Let Y a metric space, X an arbitrary set, and $\{f_n\}$ a sequence of functions from X to Y. We saw that if the f_n are bounded then f_n approaches a function $f: X \to Y$ in the $\mathcal{B}(X,Y)$ metric if and only if $f_n \to f$ uniformly. What should it mean for a sequence $\{f_n\}$ to be "uniformly Cauchy"? Prove that if Y is complete and X is a topological space then a uniformly Cauchy sequence of continuous functions from X to Y converges uniformly to a continuous function.

5. Let X, Y metric spaces, and X^*, Y^* their completions. Prove that any uniformly continuous $f: X \to Y$ extends uniquely to a continuous function $f^*: X^* \to Y^*$, and that f^* is still uniformly continuous. Show that if f is continuous, but not uniformly so, then there might not be a continuous f^* that extends f.

6. In the previous problem set we defined a metric

$$d_1(f,g) := \int_0^1 |f(x) - g(x)|\, dx$$

on the space $\mathcal{C}([0,1],\mathbf{C})$. We showed in class that $\mathcal{C}([0,1],\mathbf{C})$ is *not* complete under this metric.

i) Fix $x \in [0,1]$. Is the map $f \mapsto f(x)$ from $\mathcal{C}([0,1],\mathbf{C})$ to $\mathbf{C}$ continuous with respect to the d_1 metric?

ii) Now fix a continuous function $m: [0,1] \to \mathbf{C}$, and define a map $I_m : \mathcal{C}([0,1],\mathbf{C}) \to \mathbf{C}$ by

$$I_m(f) := \int_0^1 f(x) m(x)\, dx.$$

Prove that this map is uniformly continuous.

[By problem 5, the map I_m extends to a uniformly continuous map on the completion $L_1([0,1])$ of $\mathcal{C}([0,1],\mathbf{C})$ under the d_1 metric. This map is also linear (it satisfies the identity $I_m(af + bg) = aI_m(f) + bI_m(g)$ for all $f, g \in L_1([0,1])$ and $a, b \in \mathbf{C}$). Are there any linear maps from $L_1([0,1])$ to $\mathbf{C}$ not of that form?]

The next two problems describe, for a compact space X, certain compact spaces of functions on X or subsets of X; these are very useful in rigorous treatments of the calculus of variations and isoperimetric problems, respectively. The last problem (similar to one used a few years ago in the qualifying exam for graduate students here) is a version of the contraction mapping theorem; later in the course we'll prove and use the usual contraction mapping theorem to show existence and uniqueness of solutions of certain differential equations.

7. Let X be a compact topological space, and $\mathcal{F} \subseteq \mathcal{C}(X,\mathbf{C})$ any family of continuous functions. We say $\mathcal{F}$ is *equicontinuous* if, for each ϵ and any

$x \in X$, there exists an open set $U \subseteq X$ containing x such that

$$|f(x) - f(y)| < \epsilon$$

for all $f \in \mathcal{F}$ and $y \in U$. Prove that $\mathcal{F}$ is bounded if and only if

$$\{f(x) : x \in X, f \in \mathcal{F}\}$$

is a bounded subset of $\mathbf{C}$. Prove that $\mathcal{F}$ is totally bounded if and only if it is bounded and equicontinuous. What happens if $\mathbf{C}$ is replaced by an arbitrary complete metric space?

8. Recall that for any metric space X we gave the set $\mathcal{X}$ of nonempty, bounded, closed subsets of X the structure of a metric space using the Minkowski distance.
   i) Prove that if X is complete then so is $\mathcal{X}$.
   ii) Prove that if X is totally bounded then so is $\mathcal{X}$.

9. Let X be a metric space, and f a function from X to itself such that

$$d(x, y) > d(f(x), f(y))$$

   for all $x, y \in X$ such that $x \neq y$. Let $g : X \to \mathbf{R}$ be the real-valued function on X defined by

$$g(x) := d(x, f(x)).$$

   i) Prove that f is continuous, and has at most one fixed point (that is, there is at most one $x_0 \in X$ such that $x_0 = f(x_0)$.)
   ii) Prove that g is continuous.
   iii) Conclude that if X is compact then f has a fixed point. Must this still be true if X is complete but not necessarily compact?

Problem set is due Friday, Oct. 14, at the beginning of class. You may, however, postpone any one or two of Problems 2, 7, 8, or 9 until Oct.21, which will be the due date of the next problem set.