Math 55a: Honors Advanced Calculus and Linear Algebra

Homework Assignment #12 (12 December 2005):
Linear Algebra VIII: groups, exterior algebra, and determinants

As soon as I get into [Math 55] class, I'm fighting off a swarm
Of positive-definite non-degenerate symmetric bilinear forms!
—from a somewhat redundantly titled patter-song in *Les Phys* (P.Dong, 2001)

(In general, PDNDSBF's are probably easier to compute with than determinants and the like, but it's harder to fit "determinant" into G&S-style lyrics. . .)

Some basic facts about permutations and determinants:

1. [Using the sign homomorphism to prove familiar facts about Rubik's Cube.] Some terminology first: A permutation is called *even* or *odd* according as its sign is $+1$ or -1. If $i_1, i_2, \ldots, i_m$ are m distinct integers in $\{1, 2, \ldots, n\}$, the permutation of $\{1, 2, \ldots, n\}$ that takes i_1 to i_2, i_2 to i_3, ..., i_r to i_{r+1}, ..., i_{m-1} to i_m, and i_m back to i_1, while leaving the rest of $\{1, 2, \ldots, n\}$ fixed, is called an *m-cycle.* (In particular, the identity permutation is a 1-cycle.)
   i) Prove that an m-cycle has sign $(-1)^{m+1}$, i.e., is even iff m is odd.
   ii) Prove that no sequence of turns of Rubik's Cube can have the effect of flipping one of its edge pieces while leaving the rest unchanged.
   iii) Prove that no sequence of turns of Rubik's Cube can have the effect of switching two of its edge pieces while leaving the rest unchanged. Does this approach work for the $4 \times 4 \times 4$ Cube?

2. Let V be a finite-dimensional vector space over some field, and $A, B \in \mathcal{L}(V)$ with B of rank r. [Recall that the rank of B is the dimension of its image $B(V)$.] Prove that $\det(A + tB)$ is a polynomial in t of degree at most r.

3. Let F be a field, A the polynomial ring $F[z]$, and $P \in A$ a polynomial of degree $n > 0$. Let $V = A/PA$, an n-dimensional vector space over F (which also inherits a ring structure from A), and $T : V \to V$ the operator taking any equivalence class $[Q]$ to $[zQ]$.
   i) Determine the minimal and characteristic polynomials of T.
   ii) Assume that P is irreducible. Let $\alpha \in A$ be a polynomial not in PA. Prove that the operator on V defined by $Q \mapsto \alpha Q$ is injective, and thus invertible. Conclude that V is a field. (The fact that V is *not* a field if P is reducible is easy, as observed in class for the analogous case of $\mathbf{Z}/n\mathbf{Z}$.)

The trace and determinant of the multiplication-by-$[\alpha]$ map on V are called the *trace* and *norm* of $[\alpha]$. It's easy to see that these are respectively an F-linear functional on V and a multiplicative map from V to F.

Apropos traces and such:

4. Let V be an inner-product space with orthonormal basis $(e_1, \ldots, e_n)$. For operators $S, T \in \mathcal{L}(V)$, define
$$\langle S, T \rangle := \sum_{j=1}^{n} \langle Se_j, Te_j \rangle.$$
Prove that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{L}(V)$, that it satisfies the identity $\langle S, T \rangle = \langle T^*, S^* \rangle$, and that the inner product does not depend on the choice of orthonormal basis for V. [Thus an inner product on finite-dimensional space V yields canonically an inner product on $\mathcal{L}(V)$. Cf. also Problem 2 on PS9.]

5. Solve Exercises 16 and 19 from Chapter 10 of the textbook (pages 245 and 246).

Determinants and inner products (and another application of Gram-Schmidt):

6. i) Let $F = \mathbf{R}$ or $\mathbf{C}$, and $v_1, v_2, \ldots, v_n \in F^n$ the row vectors of an $n \times n$ matrix A. Prove that

$$|\det A| \leq \prod_{i=1}^{n} \|v_i\|$$

where $\|\cdot\|$ is the usual norm on F^n, with equality if and only if the v_i are orthogonal with respect to the corresponding inner product.

ii) Deduce that if M is a positive-definite symmetric or Hermitian $n \times n$ matrix with entries $a_{i,j}$ then

$$\det M \leq \prod_{i=1}^{n} a_{i,i},$$

with equality if and only if M is diagonal.

(We know already that $\det M$ and the diagonal entries $a_{i,i}$ are positive real numbers.)

More about exterior algebra:

7. Fix a nonnegative integer k. Let $F = \mathbf{R}$ or $\mathbf{C}$, and let V/F be a vector space equipped with an inner product $\langle \cdot, \cdot \rangle$. For $v_1, \ldots, v_k, w_1, \ldots, w_k \in V$, define

$$\langle\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle\rangle$$

to be the determinant of the $k \times k$ matrix whose (i,j) entry is $\langle v_i, w_j \rangle$. Prove that $\langle\langle \cdot, \cdot \rangle\rangle$ extends to an inner product on the exterior power $\bigwedge^k V$.

8. Let V be a finite-dimensional vector space over a field F. For $\omega \in \bigwedge^2 V$, define the *rank* of ω to be the rank of the associated alternating pairing on V^*, which in turn is the rank of the corresponding map $V^* \to V$. [If $\omega = \sum_i a_i \, (v_i \wedge w_i)$, the pairing is given by

$$\langle v^*, w^* \rangle = \sum_i a_i \, (v^*(v_i) \, w^*(w_i) - v^*(w_i) \, w^*(v_i)).]$$

In Problem 8 of the last problem set we saw in effect that this rank is an even integer, say $2k$. Use the results of that problem to give the following characterizations of k:

i) For each $m = 0, 1, 2, \ldots$, there exist $u_i, v_i \in V$ $(i = 1, \ldots, m)$ such that $\omega = \sum_{i=1}^{m} u_i \wedge v_i$ if and only if $m \geq k$. Moreover, if $m = k$ then the $2k$ vectors u_i, v_i are linearly independent.

ii) In characteristic 0 or $p > 2k$, the k-th exterior power of ω (that is, $\omega \wedge \omega \wedge \cdots \wedge \omega \in \bigwedge^{2k} V$, with k factors of ω) is nonzero, but the $(k+1)$-st exterior power vanishes.

And pfinally:

9. A square matrix A with entries a_{ij} in a field F is said to be *skew-symmetric* if its entries satisfy $a_{ij} = -a_{ji}$ for all i, j and the diagonal entries a_{ii} all vanish. If A has even order $2n$, and $n!$ is invertible in F, the *Pfaffian* $\mathrm{Pf}(A)$ can be defined thus: let $\omega \in \wedge^2(F^{2n})$ be defined by $\omega = \sum_{1 \leq i < j \leq 2n} a_{ij} \, e_i \wedge e_j$; then $\mathrm{Pf}(A) \in F$ is the scalar such that

$$\omega^n = n! \, \mathrm{Pf}(A)(e_1 \wedge e_2 \wedge \cdots \wedge e_{2n})$$

in $\wedge^{2n}(F^{2n})$. (Of course ω^n means $\omega \wedge \omega \wedge \cdots \wedge \omega$ with n factors.) Give an explicit formula for $\mathrm{Pf}(A)$ in terms of the a_{ij}, analogous to the formula for the determinant as a sum of $n!$ monomials. Prove that

$$\det(A) = (\mathrm{Pf}(A))^2.$$

What is the determinant of a skew-symmetric matrix of odd order?

This problem set is due Monday, 19 December, at the beginning of class.