Math 55a: Honors Advanced Calculus and Linear Algebra

Homework Assignment #11 (5 December 2005):
Linear Algebra VII: Fourier foretaste, and symplectic structures

It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out. — E. Artin, *Geometric Algebra.*

It is a mathematical fact that the casting of this pebble from my hand alters the centre of gravity of the universe.
— Thomas Carlyle (1795–1881), who apparently knew his John Donne ("if a clod be washed away by the sea, Europe is the less"[1]) but was not as clear on Newton's Laws...

A foretaste of Fourier analysis:

1. Let V be an infinite-dimensional real or complex inner product space. Then V has at least a countably infinite orthonormal set $\{v_n\}_{n=1}^{\infty}$ (why?). Prove that for every $v \in V$ we have $\langle v, v_n \rangle \to 0$ as $n \to \infty$. Conclude that in particular for every continuous function $f : [0,1] \to \mathbf{R}$ we have
$$\int_0^1 f(x) \cos 2\pi n x \, dx \to 0 \qquad \text{and} \qquad \int_0^1 f(x) \sin 2\pi n x \, dx \to 0$$
as $n \to \infty$.

2. Let V be the subspace of $\mathcal{C}(\mathbf{R}, \mathbf{C})$ consisting of functions $f : \mathbf{R} \to \mathbf{C}$ that are infinitely differentiable (a.k.a. "smooth": $d^n f/dx^n$ exists for $n = 1, 2, 3, \ldots$) and $\mathbf{Z}$-periodic: $f(x+m) = f(x)$ for all $x \in \mathbf{R}$ and $m \in \mathbf{Z}$. We make V into an inner-product space by defining
$$\langle f, g \rangle := \int_0^1 f(x)\overline{g(x)} \, dx.$$
Let $T : V \to V$ be the operator taking any $f \in V$ to its derivative df/dx (which again is in V).
i) Prove that T is a skew-Hermitian operator on V; that is, $T^* = -T$. What does this tell you *a priori* about the eigenvalues and eigenvectors of T?
ii) Determine the eigenvalues and eigenvectors of T. Make sure that eigenvectors with different eigenvalues are orthogonal, as we know they must be.
[Eigenvectors of an operator on a function space such as V are often called "eigenfunctions" in the literature.]

3. An $n \times n$ matrix A is said to be *circulant* when its (i,j)-th entry a_{ij} depends only on $i - j \bmod n$. Let S be the circulant matrix for which a_{ij} is 1 if $j \equiv i + 1 \bmod n$ and 0 otherwise. Show that A is circulant if and only if A is a polynomial in S. Conclude that all circulant matrices commute and are normal. What are the eigenvalues and eigenvectors of a circulant matrix with entries in $\mathbf{C}$? [As a corollary we can obtain a product formula for the determinant of any circulant matrix. Problem 3 may seem unrelated to Problems 1 and 2, but it can be viewed as part of a theory of discrete Fourier analysis.]

The remaining problems concern *symplectic spaces*, which for our purposes will be finite-dimensional vector spaces equipped with a nondegenerate, bilinear, alternating pairing. Recall that "alternating" means $\langle v, v \rangle = 0$ for all vectors v, which by the usual "polarization" trick implies $\langle w, v \rangle = -\langle v, w \rangle$ for all vectors v, w. Such pairings arise often in higher mathematics; they might not be as intuitive as inner products and other symmetric pairings, but fortunately their structure is even simpler than that of symmetric pairings. We begin by showing several ways that nondegenerate alternating pairings arise naturally, and then give the structure theorem and a few applications.

[1] From Donne's Meditation XVII, which more famously is the source of "no man is an island" and "never send to know for whom the bell tools; it tolls for thee". The Carlyle quote is from *Sartor Resartus*, according to several sources on the Web. No, this has no direct connection with linear algebra.

4. Let $\langle\cdot,\cdot\rangle$ be a sesquilinear Hermitian pairing on a complex vector space V. Prove that the imaginary part of $\langle\cdot,\cdot\rangle$ is an $\mathbf{R}$-bilinear alternating pairing on V, and is nondegenerate if and only if $\langle\cdot,\cdot\rangle$ is. (Likewise for the real part, which is symmetric rather than alternating, but that's more or less what we'd expect by now; I don't ask you to prove these analogous results for the real part because the proofs for the real and imaginary parts are so similar.)

5. Let $\pi : G \to A$ be a surjective homomorphism of groups such that A is abelian and the kernel of π is the center $Z(G)$ of G. (Recall that $Z(G)$ consists of those $z \in G$ that commute with every element of G; in particular $Z(G)$ is an abelian group.) For $g, g' \in G$ define the *commutator* $[g, g']$ by $[g, g'] = gg'g^{-1}g'^{-1}$, so $[g, g']$ is the identity if and only if g commutes with g'.
   i) Prove that $[g, g'] \in Z(G)$ for all $g, g' \in G$, and that if $\pi(g) = \pi(h)$ and $\pi(g') = \pi(h')$ then $[g, g'] = [h, h']$. Thus $[\cdot,\cdot]$ descends to a well-defined map $\langle\cdot,\cdot\rangle : A \times A \to Z(G)$.
   ii) Prove that $\langle\cdot,\cdot\rangle$ is a "perfect pairing": For each $a \in A$, the map $b \mapsto \langle a, b\rangle$ is a homomorphism $A \to Z$, which is trivial if and only if a is the identity; and for each $b \in A$, the map $a \mapsto \langle a, b\rangle$ is a homomorphism $A \to Z$, which is trivial if and only if b is the identity. Prove too that the pairing is "alternating": $\langle a, a\rangle$ is the identity for all $a \in A$, and (therefore) $\langle a, b\rangle = (\langle b, a\rangle)^{-1}$ for all $a, b \in A$.

Such G is called a "generalized Heisenberg group". We are particularly interested in the case that $Z(G)$ is a field, A is a vector space over that field, and $\langle\cdot,\cdot\rangle$ is bilinear. The next two problems give two natural examples, starting with the original Heisenberg group.

6. [Heisenberg group over a field.] Let G be the "unipotent group" of 3×3 matrices over a field F, that is, the group of upper triangular matrices all of whose diagonal entries are 1. Find $Z(G)$, show that $Z(G) \cong (F, +)$, and prove that G satisfies the hypotheses of Problem 5 with $A \cong F^2$ and the pairing $\langle\cdot,\cdot\rangle$ bilinear.

7. Now let U be a finite-dimensional vector space over $\mathbf{R}$, and let G be the set of linear transformations T_{c,u,u^*} of the space of continuous functions $f : U \to \mathbf{R}$, defined as follows: for $c \in \mathbf{R}$, $u \in U$, and $u^* \in U^*$, the operator T_{c,u,u^*} takes $f(x)$ to the function $e^{c+u^*(x)} f(x + u)$. Prove that G is a group with $Z(G) = \{T_{c,0,0} | c \in \mathbf{R}\}$, and is a generalized Heisenberg group with $A \cong U \oplus U^*$; determine $\langle\cdot,\cdot\rangle$, and check that it is in fact bilinear. When $U = \mathbf{R}$, is G isomorphic with the Heisenberg group of Problem 6?

If you remember "polarities" from Problem Set 8, you know that it is no accident that all our examples so far of symplectic spaces have even dimension. (For instance, in Problem 4 if V has dimension $n < \infty$ as a complex vector space then its dimension over $\mathbf{R}$ is $2n$.) In the next problem, you'll prove this, and moreover show that all symplectic structures on F^{2n} are equivalent. This is similar enough to what we've done with inner-product spaces that I refrain from the usual practice of pre-chewing the proof into bite-sized pieces.

8. i) Let V be a finite-dimensional vector space over a field F, and $\langle\cdot,\cdot\rangle$ be a nondegenerate alternating pairing on V. Show that V has even dimension $2n$ and a basis $(v_1, \ldots, v_n, w_1, \ldots, w_n)$ such that $\langle v_i, w_i\rangle = -\langle w_i, v_i\rangle = 1$ for each i and the pairing of any two basis vectors not of the form (v_i, w_i) or (w_i, v_i) vanishes. [Equivalently, if $v = \sum_i (a_i v_i + b_i w_i)$ and $v' = \sum_i (a'_i v_i + b'_i w_i)$ then $\langle v, v'\rangle = \sum_i (a_i b'_i - a'_i b_i)$.]
   ii) Deduce that V is isomorphic with our symplectic space $U \oplus U^*$ from Problem 7.
   iii) Obtain a theorem about antisymmetric matrices over F equivalent to (i) and (ii): an $2n \times 2n$ matrix over F is invertible and antisymmetric if and only if it is of the form [fill in the blank] for some $A \in \mathrm{GL}_{2n}(F)$. ("Antisymmetric" includes the condition that all diagonal entries vanish, even in characteristic 2 where this condition is not implied by $M^{\mathsf{T}} = -M$.)

9. Suppose now that F is a finite field of q elements. How many choices are there for the symplectic basis $(v_1, \ldots, v_n, w_1, \ldots, w_n)$? [This is also the size of the symplectic group $\mathrm{Sp}(V, \langle\cdot,\cdot\rangle) \cong \mathrm{Sp}_{2n}(F)$.]

This problem set is due Monday, 12 December, at the beginning of class. Any problems you have deferred from Problem Set 10 are still due Friday, 9 December, at the beginning of class.