Math 55a: Honors Advanced Calculus and Linear Algebra

Homework Assignment #10 (28 November 2005):
Linear Algebra VI

ЭТО НЕ ТОЛЬКО ОТРИЦАТЕЛЬНАЯ ВЕЛИЧИНА, НО ОТРИЦАТЕЛЬНАЯ ВЕЛИЧИНА ВОЗВЕДЕННАЯ В КВАДРАТ![1]

Some more results about bilinear pairings and inner products:

1. [A topological characterization of finite-dimensional inner product spaces.] Prove that the closed unit ball $\{v \in V : |v| \leq 1\}$ in an inner product space V is compact if and only if V is finite dimensional.
   [Strange as it may seem, there are situations where this result can actually be applied, for instance in the study of certain differential equations. It holds for arbitrary normed vector spaces over $\mathbf{R}$ or $\mathbf{C}$, not just inner-product spaces, but the proof is considerably harder in that generality.]

2. [Strange life in infinite-dimensional inner product spaces.] Let V be the space of continuous functions from $[0, 1]$ to $F(= \mathbf{R}$ or $\mathbf{C})$, with the usual inner product $\langle f, g\rangle := \int_0^1 f(x)\overline{g(x)}\,dx$. Let U be the subspace consisting of all functions such that $f(0) = 0$. Show that: V is **not** complete under the norm associated to the inner product; U is **not** closed in V; and $U^\perp = \{0\}$. (It follows that $V \neq U \oplus U^\perp$.) Is there a *closed* subspace $W \subset V$ such that $W \neq V$ but $W^\perp = \{0\}$? (You'll probably see next term that if $\mathcal{H}$ is any *complete* inner product space [a.k.a. Hilbert space] then $\mathcal{H} = U \oplus U^\perp$ for any closed subspace U.)

3. [Discrete subgroups of finite-dimensional real vector spaces.] Let V be a real vector space of finite dimension n, and $\{v_i | 1 \leq i \leq m\}$ a spanning set that is linearly independent **over $\mathbf{Q}$**. Thus the integer combinations $\sum_{i=1}^m a_i v_i$ $(a_i \in \mathbf{Z})$ are all distinct. Let G be the set of such linear combinations.
   i) Prove that if $m = n$ then G is a discrete subset of V (that is, each $v \in G$ has a neighborhood containing no elements of G other than v itself).
   ii) Prove that on the other hand if $m > n$ then for each $\epsilon > 0$ there exists nonzero $v \in G$ such that $|v| < \epsilon$ (so in particular G is not discrete).

4. [So when is G dense?] With V, v_i, G as in the previous problem, prove that G is dense in V if and only if for every nonzero $v^* \in V^*$ we have $v^*(v_i) \notin \mathbf{Q}$ for at least one $i \in \{1, \ldots, m\}$.
   [Hint: It may help to first observe that the topological closure $\overline{G}$ is necessarily closed under addition. This problem will likely be quite challenging even with that hint.]

5. [Semidefinite pairings.] A symmetric or Hermitian pairing $\langle\cdot,\cdot\rangle$ on an $\mathbf{R}$- or $\mathbf{C}$-vector space V is said to be *positive semidefinite* if $\langle v, v\rangle$ is a nonnegative real number for all $v \in V$. Prove that $\langle v, v\rangle = 0$ if and only if v is in the kernel of the pairing, i.e., if and only if $\langle v, w\rangle = 0$ for all $w \in V$. In particular, the set of such v is a vector subspace V_0 of V. Show that $\langle\cdot,\cdot\rangle$ yields a well-defined inner product on the quotient space V/V_0.

About normal operators:

6.–7. Solve Exercises 1, 4, 6, 7 from Chapter 7 of the textbook (page 158; here V must be an inner-product space of finite dimension). Recall that in Axler's notation the equations in Exercises 6 and 7 mean what we would write as $T(V) = T^*(V)$, $\ker T^k = \ker T$, and $T^k(V) = T(V)$.

This problem set is due Friday, 2 December, at the beginning of class.

[1] Attributed to J. V. Stalin in the article "On Sums of Squares and on Elliptic Curves over Function Fields" (*Journal of Number Theory* **3** (1971), 125–149) by J.W.S. Cassels, W.J. Ellison, and A. Pfister. I'm told that this quote translates to "Not only is this a negative quantity — it is a negative quantity squared!" I surmise that "squared" has a colloquial use in Russian comparable to the English "to the n-th degree".