Math 55a: Honors Advanced Calculus and Linear Algebra

Lemma 3.? (2 November 2005)

For any direct sum $V = \oplus_{i\in I} V_i$ of vector spaces over a field F, we have the projections $\pi_i : V \to V_i$ ($i \in I$) taking an arbitrary vector in V to its V_i component, and the embeddings $\sigma_i : V_i \to V$ taking an arbitrary vector $u \in V_i$ to the element of V with i-th coordinate u and all other coordinates zero.

For each $v \in V$, almost all the $\pi_i(v)$ are zero, and the sum of the remaining ones (more properly, of the images of the remaining ones under the σ_i) equals v. In particular, the $\pi_i(v)$ determine v. Thus if the index set I is finite we can form the linear map $\sum_{i\in I} \sigma_i \circ \pi_i : V \to V$, and note that it is the identity map on V. If I is infinite — more precisely, if $V_i \neq \{0\}$ for infinitely many $i \in I$ — the formula $\sum_{i\in I} \sigma_i \circ \pi_i = \mathbf{1}_V$ no longer makes sense *as an identity in* $\mathrm{End}(V)$: while it is true that for each $v \in V$ almost all of the summands $(\sigma_i \circ \pi_i)(v)$ vanish, this is not true of the $\sigma_i \circ \pi_i$ considered as endomorphisms of V.

Now let $T : V \to W$ be a linear map between vector spaces over the same field F. If $V = \oplus_{i\in I} V_i$, we obtain linear maps $T_i := T \circ \sigma_i : V_i \to W$. *If I is finite*, we may then compose our identity $\sum_{i\in I} \sigma_i \circ \pi_i = \mathbf{1}_V$ from the left with T to get $T = \sum_{i\in I} T_i \circ \pi_i$. This lets us recover T from the T_i. That is, the map

$$\mathrm{Hom}(V, W) \longrightarrow \bigoplus_{i\in I} \mathrm{Hom}(V_i, W) \tag{?.1}$$

taking T to $(T_i)_{i\in I} = (T \circ \sigma_i)_{i\in I}$ is a linear isomorphism, with the inverse map given by $(T_i)_{i\in I} \mapsto \sum_{i\in I} T_i \circ \pi_i$. In particular, if each $V_i \cong F$ then $|I| = \dim V$ and we obtain an isomorphism between $\mathrm{Hom}(V)$ and a direct sum of $\dim(V)$ copies of W. If W is also finite-dimensional, this yields Axler's formula (Theorem 3.20)

$$\dim(\mathrm{Hom}(V, W)) = \dim(V) \cdot \dim(W).$$

Likewise if $W = \oplus_{i\in I} W_i$ we obtain linear maps $T_i := \pi_i \circ T : V \to W_i$. *If I is finite*, we may then compose our identity $\sum_{i\in I} \sigma_i \circ \pi_i = \mathbf{1}_W$ from the right with T to get $T = \sum_{i\in I} \sigma_i \circ T_i$. As before, we deduce a linear isomorphism

$$\mathrm{Hom}(V, W) \xrightarrow{\sim} \bigoplus_{i\in I} \mathrm{Hom}(V, W_i). \tag{?.2}$$

The situation is rather more complicated if I is infinite. One thing that we can say is that if $W = \oplus_{i\in I} W_i$ and $V = F$ then we certainly have an isomorphism $\mathrm{Hom}(V, W) \cong \oplus_{i\in I} \mathrm{Hom}(V, W_i)$, because for any vector space X we have the canonical identification $T \mapsto T(1)$ of $\mathrm{Hom}(F, X)$ with X. Using (?.1) we easily deduce an isomorphism $\mathrm{Hom}(V, W) \cong \oplus_{i\in I} \mathrm{Hom}(V, W_i)$ if $\dim(V) < \infty$. This hypothesis on V cannot be dropped: if W is an infinite direct sum, and $V = W$, then we have already seen that the identity map $\mathbf{1}_W$ fails to decompose as a finite sum of maps $V \to W_i$. Nor is it true in general that $\mathrm{Hom}(V, W) \cong \oplus_{i\in I} \mathrm{Hom}(V_i, W)$, even when $W = F$. Can you give a different formula for $\mathrm{Hom}(V, W)$ in terms of the $\mathrm{Hom}(V_i, W)$?