Math 55a: Honors Advanced Calculus and Linear Algebra

Tensor products

Slogan. Tensor products of vector spaces are to Cartesian products of sets as direct sums of vectors spaces are to disjoint unions of sets.

Description. For any two vector spaces U, V over the same field F, we will construct a *tensor product* $U \otimes V$ (occasionally still known also as the "Kronecker product" of U, V), which is also an F-vector space. If U, V are finite dimensional then so is $U \otimes V$, with $\dim(U \otimes V) = \dim U \cdot \dim V$. If U has basis $\{u_i : i \in I\}$ and V has basis $\{v_j : j \in J\}$, then $U \otimes V$ has basis $\{u_i \otimes v_j : (i, j) \in I \times J\}$.

This notation $u_i \otimes v_j$ is a special case of a map $\otimes : U \times V \to U \otimes V$, which is bilinear: for each $u_0 \in U$, the map $v \mapsto u_0 \otimes v$ is a linear map from V to $U \otimes V$, and for each $v_0 \in V$, the map $u \mapsto u \otimes v_0$ is a linear map from U to $U \otimes V$. So, for instance,

$$(2u_1 + 3u_2) \otimes (4v_1 - 5v_2) = 8u_1 \otimes v_1 - 10u_1 \otimes v_2 + 12u_2 \otimes v_1 - 15u_2 \otimes v_2.$$

The element $u \otimes v$ of $U \otimes V$ is called the "tensor product of u and v".

Definition. Such an element $u \otimes v$ is called a "pure tensor" in $U \otimes V$. The general element of $U \otimes V$ is not a pure tensor; for instance you can check that

$$a_{11}u_1 \otimes v_1 + a_{12}u_1 \otimes v_2 + a_{21}u_2 \otimes v_1 + a_{22}u_2 \otimes v_2$$

is a pure tensor if and only if $a_{11}a_{22} = a_{12}a_{21}$. But any element of $U \otimes V$ is a linear combination of pure tensors. The basis-free construction of $U \otimes V$ is obtained in effect by declaring that $U \otimes V$ consists of linear combinations of pure tensors subject to the condition of bilinearity. More formally, we define $U \otimes V$ as a quotient space:

$$U \otimes V := Z/Z_0,$$

where Z is the (huge) vector space with one basis element $u \otimes v$ for every $u \in U$ and $v \in V$ (that is, Z is the space of formal (finite) linear combinations of the symbols $u \otimes v$), and $Z_0 \subseteq Z$ is the subspace generated by the linear combinations of the form

$$(u + u') \otimes v - u \otimes v - u' \otimes v, \quad u \otimes (v + v') - u \otimes v - u \otimes v',$$

$$(au) \otimes v - a(u \otimes v), \quad u \otimes (av) - a(u \otimes v)$$

for all $u, u' \in U$, $v, v' \in V$, $a \in F$.

Properties. To see this definition in action and verify that it does what we want, let us prove our claim above concerning bases: If $\{u_i\}_{i \in I}$ and $\{v_j\}_{j \in J}$ are bases for U and V then $\{u_i \otimes v_j\}$ is a basis for $U \otimes V$. Naturally, for any vectors

$u \in U,\ v \in V$, we write "$u \otimes v$" for the image of $u \otimes v \in Z$ under the quotient map $Z \to Z/Z_0 = U \otimes V$.

Let W be a vector space with basis $\{w_{ij}\}$ indexed by $I \times J$. We construct linear maps

$$\alpha : W \to U \otimes V, \quad \beta : U \otimes V \to W,$$

with $\alpha(w_{ij}) = u_i \otimes v_j$ and $\beta(u_i \otimes v_j) = w_{ij}$. We prove that α and β are each other's inverse. This will show that α, β are isomorphisms that identify w_{ij} with $u_i \otimes v_j$, thus proving our claim. In each case we use the fact that choosing a linear map on a vector space is equivalent to choosing an image of each basis vector. The map α is easy: we must take w_{ij} to $u_i \otimes v_j$. As to β, we don't yet have a basis for $U \otimes V$, so we first define a map $\tilde{\beta} : Z \to W$, and show that $Z_0 \subseteq \ker \tilde{\beta}$, so $\tilde{\beta}$ "factors through Z_0", i.e., descends to a well-defined map from $Z/Z_0 = U \otimes V$. Recall that $\{u \otimes v : u \in U,\ v \in V\}$ is a basis for Z. For all $u = \sum_i a_i u_i \in U$ and $v = \sum_j b_j v_j \in V$, we define

$$\tilde{\beta}(u \otimes v) = \sum_i \sum_j a_i b_j (u_i \otimes v_j).$$

Note that this sum is actually finite because the sums for u and v are finite, so the sum represents a legitimate element of W. We then readily see that $\ker \tilde{\beta}$ contains Z_0, because each generator of Z_0 maps to zero. We check that $\beta \circ \alpha$ and $\alpha \circ \beta$ are the identity maps on our generators of W and $U \otimes V$. The former check is immediate: $\tilde{\beta}(u_i \otimes v_j) = w_{ij}$. The latter takes just a bit more work: it comes down to showing that

$$u \otimes v - \sum_i \sum_j a_i b_j (u_i \otimes v_j) \in Z_0.$$

But this is straightforward, since the choice of $\tilde{\beta}(u \otimes v)$ was forced on us by the requirement of bilinearity. This exercise completes the proof of our claim.

Our initial Slogan, and/or the symbol $\otimes$ for tensor product, and/or the formula for $\dim(U \otimes V)$ in the finite-dimensional case, lead us to expect identities such as

$$V_1 \otimes V_2 \cong V_2 \otimes V_1, \quad (V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3),$$

and

$$U \otimes (V_1 \oplus V_2) \cong (U \otimes V_1) \oplus (U \otimes V_2).$$

These are true, and in fact are established by canonical isomorphisms taking $v_1 \otimes v_2$ to $v_2 \otimes v_1$, $(v_1 \otimes v_2) \otimes v_3$ to $v_1 \otimes (v_2 \otimes v_3)$, and $u \otimes (v_1, v_2)$ to $(u \otimes v_1, u \otimes v_2)$. In each case this is demonstrated by first defining the linear maps and their inverses on the level of the Z spaces and then showing that they descend to the tensor products which are the quotients of those Z's. Even more simply we show that

$$V \otimes F \cong V, \quad V \otimes \{0\} = \{0\}$$

for any F-vector space V.

A universal property. Suppose now that we have a linear map

$$F : U \otimes V \to X$$

for some F-vector space X. Define a function $f : U \times V \to X$ by

$$f(u, v) := F(u \otimes v).$$

Then this map is bilinear, in the sense described above. Conversely, for any function $f : U \times V \to X$ we may define $\tilde{F} : Z \to X$ by setting $\tilde{F}(u \otimes v) = f(u, v)$, and $\tilde{F}$ descends to $Z/Z_0 = U \otimes V$ if and only if f is bilinear. Thus a bilinear map on $U \times V$ is tantamount to a linear map on $U \otimes V$; more precisely, there is a canonical isomorphism between the vector space of bilinear maps: $U \times V \to X$ and the space $\mathrm{Hom}(U \otimes V, X)$ that takes f to the map $u \otimes v \mapsto f(u, v)$. Stated yet another way, every bilinear function on $U \times V$ factors through the bilinear map $(u, v) \mapsto u \otimes v$ from $U \times V$ to $U \otimes V$. This "universal property" of $U \otimes V$ could even be taken as a definition of the tensor product (once one shows that it determines $U \otimes V$ up to canonical isomorphism).

For example, a bilinear form on V is a bilinear map from $V \times V \to F$, which is now seen to be a linear map from $V \otimes V \to F$, that is, an element of the dual space $(V \otimes V)^*$. We shall come back to this important example later.

Tensor products of linear maps. Here is another key example. For any linear maps $S : U \to U'$ and $T : V \to V'$ we get a bilinear map $U \times V \to U' \otimes V'$ taking (u, v) to $S(u) \otimes T(v)$. Thus we have a linear map from $U \otimes V$ to $U' \otimes V'$. We call this map $S \otimes T$. The map $(S, T) \mapsto S \otimes T$ is itself a bilinear map from $\mathrm{Hom}(U, U') \times \mathrm{Hom}(V, V')$ to $\mathrm{Hom}(U \otimes V, U' \otimes V')$, which yields a canonical map $\mathrm{Hom}(U, U') \otimes \mathrm{Hom}(V, V') \to \mathrm{Hom}(U \otimes V, U' \otimes V')$. At least if U, U', V, V' are all finite dimensional, this map is an isomorphism. This can be seen by choosing bases for U, V, U', V'. This yields bases for $U \otimes V$ and $U' \otimes V'$ (the $u_i \otimes v_j$ construction above), for $\mathrm{Hom}(U, U')$ and $\mathrm{Hom}(V, V')$ (the matrix entries), and this for $\mathrm{Hom}(U \otimes V, U' \otimes V')$ and $\mathrm{Hom}(U, U') \otimes \mathrm{Hom}(V, V')$; and our map takes the (i, j, i', j') element of the first basis to the (i, j, i', j') element of the second. If we represent $S, T, S \otimes T$ by matrices, we get a bilinear map

$$\mathrm{Mat}(m, n) \times \mathrm{Mat}(m', n') \to \mathrm{Mat}(mm', nn')$$

called the Kronecker product of matrices; the entries of $\mathcal{M}(S \otimes T)$ are the products of each entry of $\mathcal{M}(S)$ with every entry of $\mathcal{M}(T)$.

Tensor products and duality. If the above seems hopelessly abstract, consider some special cases. Suppose $U' = V = F$. We then map $U^* \otimes V'$ to the familiar space $\mathrm{Hom}(U, V')$, and the map is an isomorphism if U, V are finite dimensional. Thus if V_i are finite dimensional then we have identified $\mathrm{Hom}(V_1, V_2)$ with $V_1^* \otimes V_2$. If instead we take $U' = V' = F$ then we get a map

$U^* \otimes V^* \to (U \otimes V)^*$, which is an isomorphism if U, V are finite dimensional. In particular, if $U = V$ we find that a bilinear form on a finite-dimensional vector space V is tantamount to an element of $V^* \otimes V^*$.

Changing the ground field. In another direction, suppose F' is a field containing F, and let $V' = V \otimes_F F'$. (When more than one field is present, we'll use the subscript to indicate the intended ground field for the tensor product. A larger ground field gives more generators for Z_0 and thus may yield a smaller tensor product Z/Z_0. In most of the applications we'll have $F = \mathbf{R}$, $F' = \mathbf{C}$.) We claim that V' is in fact a vector space over F'. For each $a \in F'$, consider multiplication by a as an F-linear map on F'. Then $1_V \otimes a$ is a linear map from V' to itself, which we use as the multiplication-by-a map on V'. The fact that multiplication by ab is the same as multiplication by b followed by multiplication by a is then a special case of the fact that composition of linear maps is consistent with tensor products:

$$(S_1 \circ S_2) \otimes (T_1 \circ T_2) = (S_1 \otimes T_1) \circ (S_2 \otimes T_2).$$

This in turn is true because it holds on pure tensors $u \otimes v$. We usually think of V' as V with scalars extended from F to F'.

If V has dimension $n < \infty$ with basis $\{v_i\}_{i=1}^n$ then $\{v_i \otimes 1\}$ is a basis for V' (to see this, begin by using $\{v_i\}$ to identify V with $F \oplus F \oplus \cdots \oplus F$, and tensor this with F'). If $T : U \to V$ is a linear map between F-vector spaces then $T \otimes 1$ is an F'-linear map from $U \otimes F = U'$ to V'; when U, V are finite dimensional, this map has the same matrix as T as long as we use the bases $\{u_i \otimes 1\}$, $\{v_j \otimes 1\}$ for U', V'. We usually think of $T \otimes 1$ as T with scalars extended from F to F'.

Symmetric and alternating tensor squares. The *tensor square* $V^{\otimes 2}$ of V is defined by

$$V^{\otimes 2} := V \otimes V.$$

Likewise we can define tensor cubes and higher tensor powers. (Of course $V^{\otimes 1}$ is V itself; what should $V^{\otimes 0}$ be?) Our isomorphism $V_1 \otimes V_2 \cong V_2 \otimes V_1$ then becomes an isomorphism s from $V \otimes V$ to itself. This map is not the identity (unless V has dimension 0 or 1), but it is always an involution; that is, s^2 is the identity. The subspace of $V \otimes V$ fixed under s is the *symmetric square* of V, denoted $\operatorname{Sym}^2 V$. It can also be defined as a quotient Z/Z_1, with Z as in the definition of $V \otimes V$, and Z_1 generated by Z_0 and combinations of the form $v_1 \otimes v_2 - v_2 \otimes v_1$. Likewise we may define the symmetric cube and higher symmetric powers of V by declaring $\operatorname{Sym}^k V$ to be the subspace of $V^{\otimes k}$ invariant under arbitrary permutations of the k indices. If V has finite dimension n then $\operatorname{Sym}^2 V$ has dimension $(n^2 + n)/2$; do you see why? What does this correspond to in terms of our motivating Slogan? Can you determine the dimension of $\operatorname{Sym}^k V$ for $k = 3, 4, \ldots$?

We can also regard $\operatorname{Sym}^2 V$ as the $+1$-eigenspace of s. Since $s^2 = 1$, we know that the only possible eigenvalues are ± 1. What then of the -1 eigenspace?

Usually this is called the *alternating square* of V, denoted by $\wedge^2 V$, and can be obtained as the quotient of Z by the subspace generated by Z_0 and combinations of the form $v_1 \otimes v_2 + v_2 \otimes v_1$; the image of $v_1 \otimes v_2$ in $\wedge^2 V$ is denoted by $v_1 \wedge v_2$. The caveat "Usually" is necessary because in characteristic 2 one cannot distinguish between -1 and $+1$! Note however that if $2 \neq 0$ then the identity $v_1 \wedge v_2 + v_2 \wedge v_1 = 0$ entails $v \wedge v = 0$ for all v. Conversely, in any characteristic the identity $v \wedge v = 0$ entails $v_1 \wedge v_2 + v_2 \wedge v_1 = 0$ for all $v_1, v_2 \in V$. In other words, the subspace Z_2 of Z generated by Z_0 and all elements of the form $v \otimes v$ contains all combinations $v_1 \otimes v_2 + v_2 \otimes v_1$. Proof:

$$v_1 \otimes v_2 + v_2 \otimes v_1 = (v_1 + v_2) \otimes (v_1 + v_2) - (v_1 \otimes v_1) - (v_2 \otimes v_2) - B,$$

where $B \in Z_0$ (why?). So, we actually define $\wedge^2 V$ to be Z/Z_2; this is identical with the -1 eigenspace of s when $2 \neq 0$, and does what we want it to even when $2 = 0$. If V has finite dimension n then $\wedge^2 V$ has dimension $(n^2 - n)/2$, and if V has basis $\{v_i\}_{i \in I}$ for some totally ordered index set I then $\wedge^2 V$ has basis $\{v_i \wedge v_j : i < j\}$. We will later define higher alternating powers $\wedge^k V$ of dimension $\binom{n}{k}$ (so $\wedge^k$ will correspond to unordered k-tuples under our Slogan). The key ingredient is the existence of the sign homomorphism from the group of permutations of $\{1, 2, \ldots, k\}$ to the two-element group $\{\pm 1\}$.

If we apply the Sym^k construction to the space V^* of linear functionals on V, we obtain the space of homogeneous polynomial functions of degree k from V to F. For instance, a *symmetric* bilinear form on V is an element of $\mathrm{Sym}^2 V^*$. Likewise $\wedge^2 V^*$ consists of the alternating (a.k.a. antisymmetric) bilinear forms on V.