1. First we will check that $d(x,y)$ is, indeed, a metric. Clearly the first axiom is satisfied from the definition of the metric. If $x=y$ then $d(x,y)=d(x,x)=d(y,x)=0$ which satisfies the second axiom. If $x\neq y$ then $d(x,y)=1=d(y,x)$ so it satisfies the second axiom. Also, $S=d(x,y)+d(y,z)$ can be one of three values: $0,1,2$. If $S=0$ then $x=y$ and $y=z$ so $x=z$ and the third axiom is satisfied. If $S=1$ or 2 then the triangle inequality is satisfied since $d(x,z)$ is at most 1. So $d(x,y)$ is, in fact, a metric.

   Now we will show that all subsets of X are both open and closed, and that the only dense set is all of X.

   Let $E\subseteq X$, and $x\in X$. Let $N=N_{1/2}(x)$. There are no points other than x in N, since all other points are at a distance 1 from it. Thus x cannot be a limit point of any subset of X. However, if $x\in E$ it is an interior point of E, since N does not contain any points not in E, so trivially lies completely inside E. Thus any subset E of X is open, since every point in it is an interior point. Also, since there are no limit points in X, E contains all of its limit points, and so E is also closed.

   Now suppose that E is dense in X, and suppose $x\in X,\ x\notin E$. Then, since N only contains x, $E\bigcap N=\emptyset$. Thus E is not dense in X, a contradiction. So there must not exist an $x\in X,\ x\notin E$. So the only subset of X that is dense in X is all of X.

2. First we will show that d_0 is a metric. Let $x,y,z\in X$. Clearly, since $d(x,y)\geq 0$ and $1+d(x,y)\geq 1$ we know that
$$\frac{d(x,y)}{1+d(x,y)}\geq 0.$$
   Also, since $d(x,y)=0$ iff $x=y$, $d_0(x,y)=0$ iff $x=y$. So d_0 satisfies the first axiom. Also,
$$d_0(x,y)=\frac{d(x,y)}{1+d(x,y)}=\frac{d(y,x)}{1+d(y,x)}=d_0(x,y).$$
   Thus d_0 satisfies the second axiom.

   Now notice that the function $f(x)=x/(1+x)$ is everywhere increasing on the positive reals. Thus if $x\geq y$ we know that $f(x)\geq f(y)$. Thus
$$\begin{aligned}\frac{d(x,y)}{1+d(x,y)}+\frac{d(y,z)}{1+d(y,z)} &= \frac{d(x,y)+d(y,z)+2d(x,y)d(y,z)}{1+d(x,y)+d(y,z)+d(x,y)d(y,z)}\\ &\geq \frac{d(x,y)+d(y,z)+d(x,y)d(y,z)}{1+d(x,y)+d(y,z)+d(x,y)d(y,z)}\\ &\geq \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)}\\ &\geq \frac{d(x,z)}{1+d(x,z)}\\ &= d_0(x,z).\end{aligned}$$
   So d_0 satisfies the third axiom, and is thus a metric.

   Let $N_{r,d}(p)$ denote a neighborhood of p with radius r under the metric d. Consider a subset E of X and a point $p\in E$. Let $M=N_{r,d}(p)$ and $M'=N_{r/(1+r),d_0}(p)$. Then if
$$x\in M\iff d(x,p)<r\iff\frac{d(x,p)}{1+d(x,p)}<\frac{r}{1+r}\iff d_0(x,p)<\frac{r}{1+r}\iff x\in M'.$$
   This shows that $M\bigcap E=M$ if and only if $M'\bigcap E=M'$, since every point is in M if and only if it is in M'. Thus a point is interior under d iff it is interior under d_0. So E is open under d iff it is open under d_0.

We know that

$$\frac{x}{1+x} = 1 - \frac{1}{1+x}.$$

In this case, x is always nonnegative (since $x = d(y,z),\ y,z \in X$) so $1/(1+x)$ is always positive. This the above fraction will always be less than 1, regardless of how large x gets. Thus the metric space (X, d_0) is bounded.

3. $d_1(x,y)$ is not a metric, since it does not satisfy the triangle inequality. Letting $x = 1, y = 2, z = 3$ we get $(1-3)^2 \le (1-2)^2 + (2-3)^2$ which is clearly false.

   $d_2(x,y)$ is a metric. It obviously satisfies axioms 1 and 2. To show axiom 3 we do the following:

$$\begin{aligned}
|x-z| &\le |x-y| + |y-z| \Rightarrow \\
|x-z| &\le |x-y| + |y-z| + 2\sqrt{|x-y||y-z|} \Leftrightarrow \\
\sqrt{|x-z|}^2 &\le (\sqrt{|x-y|} + \sqrt{|y-z|})^2 \Leftrightarrow \\
\sqrt{|x-z|} &\le \sqrt{|x-y|} + \sqrt{|y-z|}.
\end{aligned}$$

   Thus $d_2(x,y)$ is a metric.

   $d_3(x,y)$ does not satisfy axiom 1, since if $x = 1$ and $y = -1$ then $x \ne y$ but $d_3(x,y) = |1-1| = 0$.

   $d_4(x,y)$ is a metric. It trivially satisfies axioms 1 and 2. We wish to check the third axiom, $d(x,z) \le d(x,y) + d(y,z)$. Without loss of generality, $x \le z$.

   Case 1: $y < x$. Then $|x^3 - z^3| \le |x^3 - y^3| + |z^3 - y^3| \Leftrightarrow y^3 < x^3$ which is true.

   Case 2: $x \le y \le z$. Then $|x^3 - z^3| \le |x^3 - y^3| + |z^3 - y^3| \Leftrightarrow 0 \le 0$ which is also true.

   Case 3L $y > z$. Then $|x^3 - z^3| \le |x^3 - y^3| + |z^3 - y^3| \Leftrightarrow z^3 \le y^3$ which is true. So $d_4(x,y)$ satisfies the third axiom and so is a metric.

   $d_5(x,y)$ is not a metric because it does not satisfy axiom 2, since $d_5(1,2) = 3$ but $d_5(2,1) = 0$. (Incidentally, this shows it does not satisfy axiom 1, also).

   $d_6(x,y)$ is a metric. We know that $|x-y|$ is a metric, because it is the standard metric for the real numbers. From problem 2 we know that if $d(x,y)$ is a metric then so is $d(x,y)/(1 + d(x,y)$. Thus $|x-y|/(1 + |x-y|)$ is a metric.

4. (a) Clearly, $f(x) = x$ is injective, because if $f(x) = f(y)$ then trivially $x = y$. Also, for all $x \in X$ $f(x) = x$ so the map is surjective. $d(x,y) = d(f(x), f(y))$ because $f(x) = x$. Thus the identity map is always an isometry.

   (b) Suppose that $i^{-1}(x) = i^{-1}(y)$. Then, $i(i^{-1}(x)) = i(i^{-1}(y)) \Rightarrow x = y$. Thus i^{-1} is injective. Consider an $x \in X$. $i(x) \in Y$ and $i^{-1}(i(x)) = x$ so i^{-1} is surjective. Thus it is a bijection.

   Let the metric on X be d_X and the metric on Y be d_Y. Then, for $x, y \in Y$

$$d_X(i^{-1}(x), i^{-1}(y)) = d_Y(i(i^{-1}(x)), i(i^{-1}(y))) = d_Y(x,y).$$

   Thus i^{-1} is also an isometry.

   (c) Suppose $x, y \in X$ and $j(i(x)) = j(i(y))$. Then, since j is injective, $i(x) = i(y)$. But since i is injective, that means that $x = y$. So $j \circ i$ is injective. Consider $z \in Z$. Since j is surjective, we know there exists a $z' \in Y$ such that $j(z') = z$. Also, we know that there exist a $z'' \in X$ such that $i(z'') = z'$ beause i is injective. Thus $j(i(z'')) = j(z') = z$, so $j \circ i$ is surjective. Thus it is a bijection.

   Let the metrics on X, Y, Z be d_X, d_Y, d_Z, respectively. Then, for $x, y \in X$ we know that $d_X(x,y) = d_Y(i(x), y(y))$ because i is an isometry. Also, we know that $d_Y(i(x), i(y)) = d_Z(j(i(x)), j(i(y)))$, because j is an isometry. Thus we know that $d_X(x,y) = d_Z(j(i(x)), j(i(y)))$ so $j \circ i$ is an isometry.

(d) Clearly, both the functions $f(x) = -x$ and $g(x) = x + a$ are both injective and surjective, since they are linear functions. Also $|f(x) - f(y)| = |-x + y| = |y - x| = |x - y|$ so f is an isometry. $|g(x) - g(y)| = |x + a - y - a| = |x - y|$ so g is an isometry.

(e) I will show that if we know $a \in \mathbb{R}$ such that $f(a) = 0$ and we know whether $f(a + 1) = \pm 1$ the rest of the isometry is uniequly determined. Consider some $x \in \mathbb{R}$, $x \neq a, a + 1$. We know that $|f(x) - f(a)| = |x - a| \Rightarrow |f(x)| = |x - a|$. Suppose $f(a + 1) = 1$. Then we know that $|f(x) - f(a + 1)| = |x - a - 1| \Rightarrow |f(x) - 1| = |x - a - 1|$. We already know that $f(x) = a - x$ or $x - a$. Now we also know that $f(x) - 1 = x - a - 1$ or $1 + a - x$ which implies that $f(x) = x - a$ or $2 + a - x$. From this, it is clear that $f(x) = x - a$ (since for no other value of $f(x)$ do our equations agree). Analogously, if $f(a + 1) = -1$ we get that $f(x) = a - x$. Thus any isometry that maps something to zero will be of the form either $x - a$ (j_{-a}) or $a - x$ ($i \circ j_{-a}$). However, every isometry must map something to zero, since an isometry is surjective. Thus all isometries of $\mathbb{R}$ are of the form j_a or $i \circ j_a$.

5. First we will show that $\sim$ is an equivalence relation. We know that $d(p, p) = 0$ by the first axiom of a metric. Thus $p \sim p$. Also, since $d(p, q) = d(q, p)$, if $d(p, q) = 0$ then $d(q, p) = 0$, so $p \sim q$ implies $q \sim p$. Also, if $d(p, q) = 0$ and $d(q, r) = 0$, by the triangle inequality we know that $0 \leq d(p, r) \leq d(p, q) + d(q, r) = 0$ so $d(p, r) = 0$. Thus if $p \sim q$ and $q \sim r$ then $p \sim r$. Thus $\sim$ is an equivalence relation. To show that $\overline{d}([p], [q])$ is well defined we need to show that if $p \sim p'$ and $q \sim q'$ then $d(p, q) = d(p', q')$, because that would mean that for two different representatives of $[p]$ and $[q]$ the distance between them is the same, since all elements $p' \in [p]$ have $p' \sim p$ and $q' \in [q]$ has $q' \sim q$. So we apply the triangle inequality: $d(p, q) \leq d(p, p') + d(p', q') + d(q', q) = d(p', q')$. Also, $d(p', q') \leq d(p, p') + d(p, q) + d(q', q) = d(p, q)$. Since $d(p, q) \leq d(p', q')$ and $d(p', q') \leq d(p, q)$ we get that $d(p, q) = d(p', q')$. This shows that the function $\overline{d}$ is well-defined. It trivially satisfies axiom 2, since d is symmetric. For $[p] \neq [q]$ we know that $\overline{d}([p], [q]) = d(p, q)$. Thus, since $d(p, q) > 0$ (by definition of $[p]$ and $[q]$) we know that $\overline{d}([p], [q]) > 0$. Also, if $[p] = [q]$ we know that $\overline{d}([p], [q]) = d(p, q) = 0$ by definition of $[p]$ and $[q]$. So $\overline{d}$ satisfies axiom 1. Lastly, $\overline{d}([p], [q]) + \overline{d}([q], [r]) = d(p, q) + d(q, r) \geq d(p, r) = \overline{d}([p], [r])$. Thus $\overline{d}$ satisfies axiom 3. Since it satisfies all of the axioms it is a metric.

In $\mathbb{R}^3$ with the example metric, two points are at distance zero if their z-coordinates are equal. Also, their distance apart otherwise is the d_∞ metric in $\mathbb{R}^2$. Thus $\overline{\mathbb{R}^3}$ is equivalent to $\mathbb{R}^2$ with the d_∞ metric.

6. Let z be a limit point of E'. Then we know that, for $r > 0$ there exists a $y \in E'$ such that $d(y, z) < r$. Let $d(z, y) = h$. Let $r' = \min(h, r - h)$. $y \in E'$ implies that there exists $x \in E$ such that $d(y, x) < r'$. Then $d(x, z) \leq d(x, y) + d(y, z) < r' + h \leq r - h + h = r$. Thus for any $r > 0$ there exists $x \in E$ such that $d(z, x) < r$, so $z \in E'$. Thus E' is closed. If all limit points of $\overline{E}$ are limit points of either E or E' then we are done, since E' contains all of those limit points, so $\overline{E}$ would simply have E' as its limit points, which would mean that it has the same limit points as E. So suppose $\overline{E}$ has a limit point $z \notin E'$. Consider a sequence of points $\{a_n\}$ such that $d(z, a_n) < 1/n$. An infinite subsequence of these must be in either E or E', since $\overline{E} = E \bigcup E'$. But then z is a limit point of that set. However, we know that E' is the set of limit points of E and E' is closed. Thus z is in $\overline{E}$, contradicting our conjecture. Thus E and $\overline{E}$ have the same limit points.

However, E and E' do not necessarily have the same limit points. The set $A_{0,1}$ (look in 7 for definition of $A_{x,y}$) has only one limit point, 0. Thus $A'_{0,1} = 0$, which has no limit points because it is finite.

7. First I will construct a bounded subset $A_{x,y}$ of $\mathbb{R}$ that has only one limit point. $A_{x,y}$ will be a sequence of points $\{a_n\}_{n=1}^\infty$ such that

$$a_n = x + \frac{1}{2^n}|x - y|.$$

Then clearly no point outside of $[x, y)$ is a limit point. Also no point in (x, y) is a limit point, since the sequence is monotonically decreasing with limit x as $n \to \infty$. Thus every number is between two members of the sequence (which means it is not a limit point) or is a member of the sequence (and is thus also not a limit point. However, x is a limit point, since for $r > 0$, if $n = \lceil \log_2(|x - y|/r) \rceil$, $d(x, a_n) < r$.

(a) Consider the set $A = A_{0,1} \cup A_{2,3} \cup A_{4,5} \cup \{0, 2, 4\}$. From the demonstration above it is clear that this set only has three limit points. Also, the set is clearly closed.

(b) Consider the set

$$A = \bigcup_{n=0}^{\infty} A_{2^{-n}, 2^{-n-1}} \cup \bigcup_{n=0}^{\infty} \{2^{-n}\}$$

This set will be contained in the closed interval $[0, 1]$, so is bounded. Clearly, no point outside this interval is a limit point. Also, each of the intervals $[2^{-n}, 2^{-n-1})$ clearly has only one limit point, by the definition of $A_{x,y}$. Also, 0 is a limit point, since the sequence 2^{-n} converges to 0. Thus this set has an infinite, though countable, set of limit points. Clearly this set is also bounded.

8. (a) Consider any limit point z of B, and consider an infinite sequence of points a_n such that $d(z, a_n) < 1/n$. An infinite number of these must be in one of the sets A_i. Thus z will be a limit point of that A_i. Thus, since any limit point of B is in some $\overline{A_i}$ we know that $\overline{B} \subseteq \bigcup_{k=1}^{m} \overline{A_k}$. Now consider any limit point z of some A_i. We know that the sequence of points $\{a_n\}$ such that $d(z, a_n) < 1/n$ is in A_i, and is thus also in B. Thus z is a limit point of B. So $\bigcup_{k=1}^{m} \overline{A_k} \subseteq \overline{B}$. Thus $\overline{B} = \bigcup_{k=1}^{m} \overline{A_k}$.

(b) We know that B contains all of the points in each A_i. Consider a limit point z of some A_i. Since $A_i \subseteq B$ we know that z must be a limit point of B. Thus all of the points in each A_i and all of the limit points of every A_i are in $\overline{B}$, so $\bigcup_{n=0}^{\infty} \overline{A_i} \subseteq \overline{B}$.

Consider the sets $A_i = [2^{-i}, 2^{-i-1}]$. The point 0 is not contained in the closure of any of the A_i, since each is already a closed set that does not contain 0. However, the closure of their union contains the point 0.