Math 55a: Honors Advanced Calculus and Linear Algebra

Homework Assignment #7 (1 November 2002):
Linear Algebra III

The expression δ_{ij} [see below] is called the *Kronecker delta* (after the mathematician Leopold Kronecker [1823–1891], who made more substantial contributions to mathematics than this).
— Corwin and Szczarba, *Calculus in Vector Spaces*, p.124

Some basics about linear transformations and their matrices:

1.–2. Solve Exercises 6, 22, 23, 24 from Chapter 3 of the textbook (pages 59 and 61). For #6, if $S_1 \cdots S_n$ is injective, what if anything can be said of $S_1, S_2, \ldots, S_n$? For the other three exercises, note that "$\mathcal{L}(V)$" is Axler's abbreviation for "$\mathcal{L}(V, V)$" (it is also known as $\mathrm{End}(V) = \mathrm{Hom}(V, V)$).

3. Let $\mathcal{P}_n$ be the ($\mathbf{R}$- or $\mathbf{C}$-)vector space of polynomials of degree at most n, and $L : \mathcal{P}_n \to \mathcal{P}_n$ be the linear transformation taking any polynomial $P(x)$ to the polynomial
$$(L(P))(x) = (x - 3)P''(x)$$
(here P'' is the second derivative d^2P/dx^2). exhibit a matrix for L relative to a suitable basis for $\mathcal{P}_n$, and determine the kernel, image, and rank of L.

4. Let V, W be arbitrary vector spaces over the same field. Show that, for any vector v in V, the evaluation map $E_v : \mathcal{L}(V, W) \to W$ defined by $E_v(L) = L(v)$ for all $L \in \mathcal{L}(V, W)$ is a linear transformation. If V, W are finite dimensional, what is the dimension of $\ker E_v$?

5. Let V, W be vector spaces <u>over the rational field</u> $\mathbf{Q}$. Prove that a map $T : V \to W$ is linear if and only if $T(v + v') = Tv + Tv'$ for all $v, v' \in V$. (Cf. the marginal note to Exercise 2 on p.59 of the textbook.)

More about duality:

6. If $v_1, \ldots, v_n$ is a basis for V, prove that there is for each $j = 1, \ldots, n$ a unique $v_j^* \in V^*$ such that $v_j^*(v_i)$ is 1 if $i = j$ and 0 otherwise. [In other words, $v_j^*(v_i) = \delta_{ij}$, the "Kronecker delta" referred to above, which is also the (i, j) entry of the identity matrix.] Show further that the v_j^* constitute a basis for V^*. This is called the *dual basis* to $(v_1, \ldots, v_n)$.

7. We saw that, for any vector spaces V, W, the dual of $V \oplus W$ is naturally identified with $V^* \oplus W^*$. What is the dual of $\oplus_{i \in I} V_i$? Use this to construct a vector space V over some field F such that V is <u>not</u> isomorphic with V^*.

8. Let $x_0, \ldots, x_m$ be distinct elements of F. Recall that the $m + 1$ vectors $v_i := (x_0^i, x_1^i, \ldots, x_m^i)$ $(0 \le i \le m)$ constitute a basis of F^{m+1}. Describe the dual basis.

Problem set is due Friday, Nov. 8 in class.