Math 55a: Honors Advanced Calculus and Linear Algebra

Homework Assignment #3 (4 October 2002):
Sequences cont'd; compactness start'd

Q: What did the mathematician say as ϵ approached zero?
A: "There goes the neighborhood."
—*Hoary math joke*

More about sequences and $\mathcal{C}(X, Y)$:

1. Prove that[1]
$$d_1(f,g) := \int_0^1 |f(x) - g(x)|\,dx$$
is a metric on the space $\mathcal{C}([0,1], \mathbf{C})$ of (bounded) continuous functions $f : [0,1] \to \mathbf{C}$ on the closed unit interval $[0,1]$.

2. Define sequences $\{f_n\}$, $\{g_n\}$ $(n = 1, 2, 3, \ldots)$ of functions from $\mathbf{R}$ to $\mathbf{R}$ by
$$f_n(x) = \frac{n}{x^2 + n^2}, \qquad g_n(x) = \frac{n^2}{x^2 + n^2}.$$
i) Do these sequences of functions f_n and g_n converge pointwise?
ii) Do they converge uniformly on $\mathbf{R}$? Explain.

More about separation properties in metric spaces and general topological spaces:

3. Let X be a metric space and A, B disjoint closed subsets. Prove that there exist disjoint open sets U, V such that $U \supseteq A$ and $V \supseteq B$.

4. A topological space X is said to be *normal* if it is Hausdorff and has the property that for every disjoint closed subsets A, B there exist disjoint open sets U, V such that $U \supseteq A$ and $V \supseteq B$. Show that if A, B are disjoint closed subsets of a normal space X then there exists a continuous function $f : X \to \mathbf{R}$ such that $f(X) \subseteq [0,1]$ and $f(x) = 0$ if $x \in A$ while $f(x) = 1$ if $x \in B$.
[Hint: If $S \subset [0,1]$ is a dense subset then a function $f : X \to [0,1]$ is specified completely by the sets $f^{-1}([s,1])$ for $s \in S$.]

To put #4 in context: the distance function on a metric space gives us a ready source of continuous functions from the space to $\mathbf{R}$; in particular, enough such functions to "separate points": if $x_0 \neq x_1$ then there's a continuous function taking x_0 to 0 and x_1 to 1. [For instance, the function $x \mapsto d(x, x_0)/d(x_1, x_0)$ does the trick.] According to Problem 4, $\mathcal{C}(X, \mathbf{R})$ separates points also under the hypothesis of normality, which is weaker than (well, at least as weak as) metrizability by #3.

[1] Yes, I know: we have yet to define $\int_0^1$. For this problem, though, only the most basic facts are needed, such as the existence of $\int_0^1 F(x)\,dx$ for any continuous function $F : [0,1] \to \mathbf{R}$, and the fact that if $F(x) \leq G(x)$ for all $x \in [0,1]$ then $\int_0^1 F(x)\,dx \leq \int_0^1 G(x)\,dx$.

Problem 4 should be quite challenging. By contrast, problems 5–7, concerning alternative formulations of compactness, should be fairly routine. Use these problems to also practice good solution writing. Especially with concepts such as continuity and compactness with several equivalent definitions, one can easily fall into the habit of plowing ahead with the first approach that comes to mind, which may produce a correct but rather unenlightening solution. As an extreme example, having learned that a product of two continuous functions is continuous, one could still demonstrate the continuity of a product of three continuous functions with an ϵ-δ proof; such a solution, if correct, will earn you full marks but little sympathy. A more elegant solution has the long-term advantages of deeper understanding as well as ease of review as final exams approach, and the short-term advantages of being less error prone and easier on Andrei to grade.

5. A family $\mathcal{F}$ of subsets of a set X is said to have the *finite intersection property* if $F_1 \cap \ldots \cap F_n \neq \emptyset$ for any n and $F_1, \ldots, F_n \in \mathcal{F}$ (i.e. finite intersections in $\mathcal{F}$ are nonempty). Prove that a topological space is compact if and only if $\cap_{F \in \mathcal{F}} F \neq \emptyset$ for every family $\mathcal{F}$ of closed subsets of X with the finite intersection property.

6. Show directly that a sequentially compact subset of a metric space is closed and totally bounded.

7. Say that a subset E of a metric space X is "totally bounded relative to X" if, for each $\epsilon > 0$, there is a finite cover of E by ϵ-neighborhoods in X. Prove that E is totally bounded relative to X if and only if E is totally bounded. [That is, allowing centers of the ϵ-neighborhoods to be in a larger ambient metric space does not change the notion of total boundedness.]

8. [Lebesgue Covering Lemma] Let $\{U_\alpha\}$ be an open cover of the compact metric space X. Show that there exists $r > 0$ such that, for every $x \in X$, the r-ball $B_r(x)$ is contained in some U_α. [Such r is a "Lebesgue number" for the open cover. Proceed by contradiction, assuming no such r exists. Construct a sequence $\{x_n\}$ in X such that $B_{1/n}(x_n)$ is contained in no U_α. Let x be the limit of a convergent subsequence. Show that $B_\rho(x) \subseteq U_\alpha$ for some $\rho > 0$ and α. Obtain the contradiction by showing that $B_\rho(x)$ contains some $B_{1/n}(x_n)$.]

Problem set is due Friday, Oct. 11, at the beginning of class.