Math 55a: Honors Advanced Calculus and Linear Algebra

Homework Assignment #2 (27 September 2002):
Metrics, topology, continuity, and sequences

> **Sketch of a proof** *n.* I couldn't verify all the details, so I'll break it down into the parts I couldn't prove.[1]

Please avoid merely "sketching" (as defined in the above quote) a proof. In all problem sets, you may use the result in one problem (or problem part) to solve another, even if you have not proved the first one, unless this becomes circular [EXCEPTION: when problem B is clearly a generalization of A, don't use B to solve A unless you've solved B!]. NB the problems are generally *not* in order of difficulty. Problem set is due Friday, Oct. 4, at the beginning of class.
Two different notions of distance between subsets of a metric space:

1. [Distance between subsets of a metric space] For any two subsets A, B of a metric space X, define the *distance* $d(A, B)$ between A and B by

$$d(A, B) := \inf\{d(x, y) : x \in A, y \in B\}.$$

Prove that for any subsets A, B, C of X and any element $x \in X$ we have:
i) $d(\bar{A}, \bar{B}) = d(A, B)$ (where $\bar{A}, \bar{B}$ are the closures of A, B respectively);
ii) $d(\{x\}, A) = 0$ if and only if $x \in \bar{A}$;
iii) $d(A, B \cup C) = \min\{d(A, B), d(A, C)\}$;
iv) $d(A, \{x\}) + d(\{x\}, B) \geq d(A, B)$.
Must the triangle inequality $d(A, C) + d(C, B) \geq d(A, B)$ also hold?

2. [Minkowski distance between nonempty bounded closed subsets] For a subset A of a metric space X, and a positive real number r, define

$$N_r(A) := \bigcup_{x \in A} N_r(x).$$

(Recall that $N_r(x)$ is the radius-r neighborhood of x, a.k.a. the open ball of radius r about x; one may visualize $N_r(A)$ as the radius-r neighborhood of A. For instance, $N_r(\emptyset) = \emptyset$; $N_r(\{x\}) = N_r(x)$; $N_r(X) = X$; and $r' \geq r \Rightarrow N_{r'}(A) \supseteq N_r(A)$.) For two *nonempty, bounded, closed* subsets A, B of a metric space X, define the *Minkowski distance* $\delta(A, B)$ between A and B by

$$\delta(A, B) := \inf\{r : N_r(A) \supseteq B \text{ and } N_r(B) \supseteq A\}.$$

Prove that this defines a metric on the space of nonempty, bounded, closed subsets of X.

More about the topology of $\mathbf{R}$, and relation with continuity:

3. Prove that the only subsets of $\mathbf{R}$ that are simultaneously open and closed are $\emptyset$ and $\mathbf{R}$.

[1] *Definitions of Terms Commonly Used in Higher Math*, R. Glover et al.

4. Suppose X, Y are metric spaces, and that X has the discrete metric. Find all continuous maps from X to Y. Find all continuous maps from $\mathbf{R}$ to X.

Some more topological notions:

5. A topological space is said to be *Hausdorff* if, for any two distinct elements p, q of the space, there are disjoint open sets U, V with $U \ni p$ and $V \ni q$. For instance, a metric space is automatically Hausdorff, since we may take U and V to be the open balls of radius $\frac{1}{2}d(p,q)$ about p and q.
   i) Prove that in a Hausdorff space every single-point set is closed.
   ii) Now let X, Y be topological spaces with Y Hausdorff, and let f, g be any continuous functions from X to Y. If $S \subset X$ is a dense subset such that $f(s) = g(s)$ for all $s \in S$, prove that $f = g$, i.e., that $f(x) = g(x)$ for all $x \in X$. [Naturally you must use the topological definition of denseness: "S is dense in X" means that the only open set in X disjoint from S is $\emptyset$.]

6. [Non-metrizable topologies] Recall that a *topology* on a set X is a family $\mathcal{T}$ of subsets of X which contains $\emptyset$, X, and the finite intersection and arbitrary union of any sets in $\mathcal{T}$. We noted that the open sets in a metric space constitute a topology, but not all topologies arise in this way; for instance, for any set X with more than 1 element, $\{\emptyset, X\}$ is a non-metric topology, because in a metric topology all one-point sets are closed. Suppose now that $\mathcal{T}$ is a non-metric topology on X containing all complements of one-point sets (so that all one-point sets are closed). Show that X is infinite, and construct such a topology on a countably infinite set.

7. [Homeomorphism] A *homeomorphism* between two topological spaces[2] X, Y is a bijection $f : X \to Y$ such that both f and the inverse function $f^{-1} : Y \to X$ are continuous. Show that a bijection $f : X \to Y$ is a homeomorphism if and only if f identifies the topologies of X and Y, i.e., the open sets of Y are precisely the images of open sets of X. Two topological spaces X, Y are said to be *homeomorphic* if there is a homeomorphism between them. Prove that this is an equivalence relation. Show that any isometry is a homeomorphism. Prove that every open ball in $\mathbf{R}$ is homeomorphic with $\mathbf{R}$ but not isometric with $\mathbf{R}$. (Warning: for this last part it is not enough to exhibit a non-isometric homeomorphism; you must show that <u>no</u> bijection between the ball and $\mathbf{R}$ is an isometry.)

Convergence and sequences:

8. [Rudin, p.78, Exercise 1] Suppose $s_n \in \mathbf{R}$. Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?

9. [Another characterization of convergence] Let E be the subset $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ of $\mathbf{R}$. A sequence $\{s_n\}$ in an arbitrary metric space X is equivalent to the map $\tilde{s} : E \to X$ that takes $1/n$ to s_n. Show that $\bar{E} = E \cup \{0\}$, and prove that $\{s_n\}$ converges if and only if $\tilde{s}$ extends to a continuous function on $\bar{E}$.

[2] Naturally a "topological space" is a set X endowed with a topology $\mathcal{T}$ of subsets of X.