Math 55a: Honors Advanced Calculus and Linear Algebra

Homework Assignment #11 (9 December 2002):
Linear Algebra VII

As soon as I get into [Math 55] class, I'm fighting off a swarm
Of positive-definite non-degenerate symmetric bilinear forms!
—from a somewhat redundantly titled patter-song in *Les Phys* (P.Dong, 2001)

(In general, PDNDSBF's are probably easier to compute with than determinants and the like, but it's harder to fit "determinant" into G&S-style lyrics...)

A foretaste of Fourier analysis:

1. Let V be an infinite-dimensional real or complex inner product space. Then V has at least a countably infinite orthonormal set $\{v_n\}_{n=1}^{\infty}$ (why?). For any $v \in V$ prove that $\langle v, v_n \rangle \to 0$ as $n \to \infty$. Conclude that in particular for any continuous function $f : [0,1] \to \mathbf{R}$ we have

$$\int_0^1 f(x) \cos 2\pi n x \, dx \to 0 \qquad \text{and} \qquad \int_0^1 f(x) \sin 2\pi n x \, dx \to 0$$

as $n \to \infty$.

2. Let V be the subspace of $\mathcal{C}(\mathbf{R}, \mathbf{C})$ consisting of functions $f : \mathbf{R} \to \mathbf{C}$ that are infinitely differentiable (a.k.a. "smooth": $d^n f/dx^n$ exists for $n = 1, 2, 3, \ldots$) and $\mathbf{Z}$-periodic: $f(x+m) = f(x)$ for all $x \in \mathbf{R}$ and $m \in \mathbf{Z}$. We make V into an inner-product space by defining

$$\langle f, g \rangle := \int_0^1 f(x) \overline{g(x)} \, dx.$$

Let $T : V \to V$ be the operator taking any $f \in V$ to its derivative df/dx (which again is in V).

i) Prove that T is a skew-Hermitian operator on V; that is, $T^* = -T$. What does this tell you *a priori* about the eigenvalues and eigenvectors of T?

ii) Determine the eigenvalues and eigenvectors of T. Make sure that eigenvectors with different eigenvalues are orthogonal, as we know they must be.
[Eigenvectors of an operator on a function space such as V are often called "eigenfunctions" in the literature.]

3. An $n \times n$ matrix A is said to be *circulant* when its i,j-th entry a_{ij} depends only on $i - j \bmod n$. Let S be the circulant matrix for which a_{ij} is 1 if $j \equiv i + 1 \bmod n$ and 0 otherwise. Show that A is circulant if and only if A is a polynomial in S. Conclude that all circulant matrices commute and are normal. What are the eigenvalues and eigenvectors of a circulant matrix with entries in $\mathbf{C}$? [As a corollary we can obtain a product formula for the determinant of any circulant matrix. Problem 3 may seem unrelated to Problems 1 and 2, but it will turn out to be part of a theory of discrete Fourier analysis.]

Some basic facts about permutations and determinants:

4. [Using the sign homomorphism to prove familiar facts about Rubik's Cube.] Some terminology first: A permutation is called *even* or *odd* according as its sign is $+1$ or -1. If $i_1, i_2, \ldots, i_m$ are m distinct integers in $\{1, 2, \ldots, n\}$, the permutation of $\{1, 2, \ldots, n\}$ that takes i_1 to i_2, i_2 to i_3, $\ldots$, i_r to i_{r+1}, $\ldots$, i_{m-1} to i_m, and i_m back to i_1, while leaving the rest of $\{1, 2, \ldots, n\}$ fixed, is called an *m-cycle*. (In particular, the identity

permutation is a 1-cycle.)

i) Prove that an m-cycle has sign $(-1)^{m+1}$, i.e., is even iff m is odd.

ii) Prove that no sequence of turns of Rubik's Cube can have the effect of flipping one of its edge pieces while leaving the rest unchanged.

iii) Prove that no sequence of turns of Rubik's Cube can have the effect of switching two of its edge pieces while leaving the rest unchanged. Does this approach work for the $4 \times 4 \times 4$ Cube?

5. Let V be a finite-dimensional vector space over some field, and $A, B \in \mathcal{L}(V)$ with B of rank r. [Recall that the rank of B is the dimension of its image $B(V)$.] Prove that $\det(A + tB)$ is a polynomial in t of degree at most r.

6. Let F be a field, A the polynomial ring $F[z]$, and $P \in A$ a polynomial of degree $n > 0$. Let $V = A/PA$, an n-dimensional vector space over F (which also inherits a ring structure from A), and $T : V \to V$ the operator taking any equivalence class $[Q]$ to $[zQ]$.

i) Determine the minimal and characteristic polynomials of T.

ii) Assume that P is irreducible. Let $\alpha \in A$ be a polynomial not in PA. Prove that the operator on V defined by $Q \mapsto \alpha Q$ is injective, and thus invertible. Conclude that V is a field. (The fact that V is *not* a field if P is reducible is easy, as observed in class for the analogous case of $\mathbf{Z}/n\mathbf{Z}$.)

The trace and determinant of the multiplication-by-$[\alpha]$ map on V are called the *trace* and *norm* of $[\alpha]$. It's easy to see that these are respectively an F-linear functional on V and a multiplicative map from V to F.

Apropos traces and such:

7. Let V be an inner-product space with orthonormal basis $(e_1, \ldots, e_n)$. For operators $S, T \in \mathcal{L}(V)$, define

$$\langle S, T \rangle := \sum_{j=1}^{n} \langle Se_j, Te_j \rangle.$$

Prove that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{L}(V)$, that it satisfies the identity $\langle S, T \rangle = \langle T^*, S^* \rangle$, and that the inner product does not depend on the choice of orthonormal basis for V. [Thus an inner product on finite-dimensional space V yields canonically an inner product on $\mathcal{L}(V)$. Cf. also Problem 2 on PS9.]

8. Solve Exercises 16 and 19 from Chapter 10 of the textbook (pages 245 and 246).

And pfinally:

9. A square matrix A with entries a_{ij} in a field F is said to be *skew-symmetric* if its entries satisfy $a_{ij} = -a_{ji}$ for all i, j and the diagonal entries a_{ii} all vanish. If A has even order $2n$, and $n!$ is invertible in F, the *Pfaffian* $\mathrm{Pf}(A)$ can be defined thus: let $\omega \in \wedge^2(F^{2n})$ be defined by $\omega = \sum_{1 \le i < j \le 2n} a_{ij}\, e_i \wedge e_j$; then $\mathrm{Pf}(A) \in F$ is the scalar such that

$$\omega^n = n!\, \mathrm{Pf}(A)(e_1 \wedge e_2 \wedge \cdots \wedge e_{2n})$$

in $\wedge^{2n}(F^{2n})$. (Of course ω^n means $\omega \wedge \omega \wedge \cdots \wedge \omega$ with n factors.) Give an explicit formula for $\mathrm{Pf}(A)$ in terms of the a_{ij}, analogous to the formula for the determinant as a sum of $n!$ monomials. Prove that

$$\det(A) = (\mathrm{Pf}(A))^2.$$

What is the determinant of a skew-symmetric matrix of odd order?

This problem set is due Wednesday, 18 December, at the beginning of class.