Math 55a: Honors Advanced Calculus and Linear Algebra

Homework Assignment #1 (20 September 2002):
Metric Topology

"I'm sorry..."

"Don't topologize."

—Martin Gardner (adapted)

Definition and constructions of metric spaces:[1]

1. [Cf. Rudin, p.44, Ex.10] For any set X define the *discrete metric* on X by $d(p, q) = 0$ if $p = q$ and $d(p, q) = 1$ if $p \neq q$. Prove that this is indeed a metric. With this metric, which subsets of X are open? Which are closed? Which are dense?

2. Let (X, d) be a metric space. Define $d_0(x, y) := d(x, y)/(1 + d(x, y))$ for all $x, y \in X$. Prove that d_0 is also a metric on X. Prove that a subset of X is open under the metric d if and only if it is open under d_0. [Thus (X, d) and (X, d_0) are the same as "topological spaces", but generally[2] not as metric spaces.] Show that the metric space (X, d_0) is always bounded, even though (X, d) may not be.

3. [Cf. Rudin, p.44, Ex.11] Which of the following defines a metric on $\mathbf{R}$? Explain.
   i) $d_1(x, y) := (x - y)^2$
   ii) $d_2(x, y) := \sqrt{|x - y|}$
   iii) $d_3(x, y) := |x^2 - y^2|$
   iv) $d_4(x, y) := |x^3 - y^3|$
   v) $d_5(x, y) := |x - 2y|$
   vi) $d_6(x, y) := |x - y|/(1 + |x - y|)$

4. Prove that:
   i) The identity map on a metric space is always an isometry.
   ii) If $i : X \to Y$ is an isometry, then so is the inverse map $i^{-1} : Y \to X$.
   iii) If $i : X \to Y$ and $j : Y \to Z$ are isometries, so is the composite map $j \circ i : X \to Z$.

[Note that $j \circ i$ is the correct order, not $i \circ j$. One sometimes expresses facts (i) and (iii) by saying that metric spaces and isometries between them form a "category". Parts (i),(ii),(iii) together, applied in the special case $X = Y = Z$, are expressed by saying that the isometries from X to itself constitute a "group". The remaining two parts study this group in the special case of the metric space $\mathbf{R}$.]

[1] NB: Parts of these problems refer to notions that will not be developed in class until Monday.

[2] When are (X, d) and (X, d_0) actually isometric (identical as metric spaces)?

iv) For $X = Y = \mathbf{R}$, the function $i(x) = -x$ is an isometry, as is $j_a(x) = x+a$ for any $a \in \mathbf{R}$.

v) Every isometry from $\mathbf{R}$ to itself is either j_a or $i \circ j_a$ for some a. (This last is by far the hardest part of this problem; some mathematicians would say — *after* solving the problem — "the only nontrivial" instead of "by far the hardest"...)

5. Suppose X is a set and $d : X \times X \to \mathbf{R}$ is a function satisfying all the distance axioms except $d(p, q) = 0 \Rightarrow p = q$. For example, we may take $X = \mathbf{R}^3$ and

$$d\big((x_1, x_2, x_3), (x_1', x_2', x_3')\big) := \max(|x_1 - x_1'|, |x_2 - x_2'|).$$

(Check that this is in fact an example.) For $p, q \in X$ define $p \sim q$ to mean $d(p, q) = 0$. Prove that this is an *equivalence relation*: $p \sim p$ for all $p \in X$, $p \sim q \Rightarrow q \sim p$, and $p \sim q$, $q \sim r \Rightarrow p \sim r$ [Rudin, Definition 2.3, p.25]. Show that if $p \sim p'$ and $q \sim q'$ then $d(p, q) = d(p', q')$. Let $\tilde{X}$ be the set of *equivalence classes*, i.e., subsets of X of the form $[p]$, defined as $[p] := \{p' \in X : p \sim p'\}$. [NB $[p] = [p'] \iff p \sim p'$.] Show that

$$\tilde{d}([p], [q]) = d(p, q)$$

is a well-defined function[3] on $\tilde{X} \times \tilde{X}$, and satisfies the axioms of a metric. Thus $\tilde{X}$ becomes a metric space. What is this metric space for our above example with $X = \mathbf{R}^3$?

Closures, etc.:

6. [Rudin, p.43, Ex.6] Let E be a subset of a metric space, and E' its set of limit points. Prove that E' is closed, and that E and $\bar{E}$ have the same limit points. (Recall that $\bar{E}$, the *closure* of E, is defined by $\bar{E} = E \cup E'$.) Must E and E' have the same limit points?

7. [Rudin, p.43-4, Ex.5,13]

   i) Construct a bounded closed subset of $\mathbf{R}$ with exactly three limit points.

   ii) [This is rather trickier] Construct a bounded closed set $E \subset \mathbf{R}$ for which E' is an (infinite) countable set.

8. [Rudin, p.43, Ex.7] Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.

   i) If $B_n = \cup_{i=1}^n A_i$, prove that $\bar{B}_n = \cup_{i=1}^n \bar{A}_i$ (the closure of a finite union is the union of the closures).

   ii) If $B = \cup_{i=1}^\infty A_i$, prove that $\bar{B} \supseteq \cup_{i=1}^\infty \bar{A}_i$ (the closure of a countable union contains the union of the closures). Show, by an example, that this inclusion may be proper (a.k.a. strict).

This problem set is due Friday, 27 September, at the beginning of class.

[3] That is, show that for any $P, Q \in \tilde{X}$ the value of $d(P, Q)$ does not depend on the choice of representatives of the equivalence classes P, Q.

The following problems, concerning two different notions of distance between subsets of a metric space, will be on the next problem set; if the above was a breeze you might want to take them on now:

A. [Distance between subsets of a metric space] For any two subsets A, B of a metric space X, define the *distance* $d(A, B)$ between A and B by

$$d(A, B) := \inf\{d(x, y) : x \in A, y \in B\}.$$

Prove that for any subsets A, B, C of X and any element $x \in X$ we have:
i) $d(\bar{A}, \bar{B}) = d(A, B)$ (where $\bar{A}, \bar{B}$ are the closures of A, B respectively);
ii) $d(\{x\}, A) = 0$ if and only if $x \in \bar{A}$;
iii) $d(A, B \cup C) = \min\{d(A, B), d(A, C)\}$;
iv) $d(A, \{x\}) + d(\{x\}, B) \geq d(A, B)$.
Must the triangle inequality $d(A, C) + d(C, B) \geq d(A, B)$ also hold?

B. [Minkowski distance between nonempty bounded closed subsets] For a subset A of a metric space X, and a positive real number r, define

$$N_r(A) := \bigcup_{x \in A} N_r(x).$$

[Recall that $N_r(x)$ is the radius-r neighborhood of x, a.k.a. open ball of radius r about x; one may visualize $N_r(A)$ as the radius-r neighborhood of A. For instance, $N_r(\emptyset) = \emptyset$; $N_r(\{x\}) = N_r(x)$; $N_r(X) = X$; and $r' \geq r \Rightarrow N_{r'}(A) \supseteq N_r(A)$.] For two *nonempty, bounded, closed* subsets A, B of a metric space X, define the *Minkowski distance* $\delta(A, B)$ between A and B by

$$\delta(A, B) := \inf\{r : N_r(A) \supseteq B \text{ and } N_r(B) \supseteq A\}.$$

Prove that this defines a metric on the space of nonempty, bounded, closed subsets of X.