Math 55a: Honors Advanced Calculus and Linear Algebra

The Fundamental Theorem of Algebra

A little field algebra. Let F be any field, and P a polynomial in one variable T with coefficients in F (the ring of such polynomials is usually denoted by $F[T]$). For any $t_1 \in F$, polynomial division yields a polynomial $P_1 \in F[T]$ and scalar $c \in F$ such that $P(T) = (T - t_1)P_1(t) + c$; moreover, substituting t_1 for T, we find that the "remainder" c is just $P(t_1)$. In particular, $P(t_1) = 0$ if and only if $P(T) = (T - t_1)P_1(t)$ for some polynomial P_1. Recall that a solution of the polynomial equation $P(t) = 0$ is called a <u>root</u> of P; so we are saying that t_1 is a root of P if and only if $P(T) = (T - t_1)P_1(t)$ for some polynomial P_1.

Assume now that P is not identically zero [i.e., not the zero element of $F[T]$]. We say that P *factors completely* if there exist $a, t_1, \ldots, t_n \in F$ such that

$$P(T) = a \prod_{j=1}^{n} (T - t_j).$$

Necessarily $a \neq 0$ and $n = \deg P$. Then $P(t) = 0$ if and only if $t = t_j$ for some j; i.e., the roots of P are the t_j. This gives us the implication (i)$\Rightarrow$(ii) in our next result:

Proposition. *The following conditions on a field F are equivalent:*
i) Every nonzero polynomial in $F[t]$ factors completely.
ii) Every nonconstant polynomial in $F[t]$ has a root in F.

Proof: It remains to prove (ii)$\Rightarrow$(i). We do this by induction on the degree n of our polynomial. The case $n = 0$ is trivial: then our polynomial is a constant, and we may let a be that constant. Suppose then that we have done the case of degree $n = d$, and let P be a polynomial of degree $d + 1$. Since $d + 1 > 0$, this polynomial is nonconstant, so by (ii) it has a root t_1. But then $P(T) = (T - t_1)P_1(t)$ for some polynomial P_1 of degree d. By the inductive hypothesis, P_1 factors completely; thus so does P. $\square$

A field is said to be *algebraically closed* if satisfies the two equivalent conditions of this proposition.

The case $F = \mathbf{C}$. The *Fundamental theorem of algebra* is the statement that the field $\mathbf{C}$ of complex numbers is algebraically closed. That is, every nonconstant $P \in \mathbf{C}[T]$ has a complex root. We prove this by contradiction. Suppose, then, that P is a nonconstant polynomial with no complex roots; thus P maps $\mathbf{C}$ to $\mathbf{C}^*$. Write $P(T) = \sum_{k=0}^{n} a_k T^k$ with a_n nonzero; we may also assume a_0 nonzero (why?). The idea of the proof is to look at how many times $P(t)$ winds around the origin as t moves around the circle of radius r. For small r, we expect that $P(t)$ moves in a small neighborhood of a_0, so the winding number is zero; for large r, we expect that $P(t)$ is dominated by $a_k t^k$, which winds around the origin n times. But $n \neq 0$, and the winding number shouldn't change as r increases from "small" to "large".

To make a proof out of this, we need to formalize the notion of "winding number" in this setting, and study its properties. You will encounter many of these ideas again in greater generality when you study algebraic topology.

For $\alpha \in \mathbf{R}$, define $e(\alpha) = \cos 2\pi\alpha + i \sin 2\pi\alpha$; this is a complex number of absolute value 1, with $e(\alpha) = e(\beta) \iff \alpha \equiv \beta \bmod \mathbf{Z}$. Every nonzero complex number z can be written as $|z|e(\alpha)$ with $\alpha \in \mathbf{R}$ uniquely determined mod $\mathbf{Z}$. [You'll recognize $(|z|, 2\pi\alpha)$ as the "polar coordinates" of z.] Moreover α is a continuous function from $\mathbf{C}^*$ to $\mathbf{R}/\mathbf{Z}$; thus $\alpha \circ P : \mathbf{C} \to \mathbf{R}/\mathbf{Z}$ is also continuous. For each $r > 0$, the complex numbers t with $|t| = r$ also constitute a circle homeomorphic with $\mathbf{R}/\mathbf{Z}$. So we are in effect studying continuous maps from the circle $\mathbf{R}/\mathbf{Z}$ to itself. We shall attach to such a map an integral invariant called its "winding number", and show that it is locally constant on $\mathcal{C}(\mathbf{R}/\mathbf{Z}, \mathbf{R}/\mathbf{Z})$. We shall then prove that the winding number of $a \mapsto \alpha(P(re(a)))$ is 0 for small r and n for large r, which will complete our proof of the Fundamental Theorem of Algebra.

First let $f \in \mathcal{C}([0,1], \mathbf{R}/\mathbf{Z})$. We shall prove: there exists $\tilde{f} : \mathcal{C}([0,1], \mathbf{R})$ such that $f(0) = 0$ and $\tilde{f}(a) \equiv f(a) - f(0) \bmod \mathbf{Z}$ for all $a \in [0,1]$; one can think of $\tilde{f}(a)$ as measuring the net local change of f from 0 to a. Uniqueness is easy: the difference between two such functions is a continuous map from $[0,1]$ to $\mathbf{Z}$, which is thus constant; since the functions agree at $a = 0$, they must agree throughout. For existence, use compactness of $[0,1]$ to find $\delta > 0$ such that $|a-b| \leq \delta$ implies $|f(a) - f(b)| < 1/4$, and inductively construct f on $[0, n\delta]$ for $n = 1, 2, 3, \ldots, \lfloor \delta^{-1} \rfloor$, and finally on all of $[0,1]$. *(This sketch will be filled in in class.)*

So we have a function $f \mapsto \tilde{f}$ from $\mathcal{C}([0,1], \mathbf{R}/\mathbf{Z})$ to $\mathcal{C}([0,1], \mathbf{R})$. We next claim that this function is continuous. Indeed if for some $d(f,g) < \epsilon$ for some $\epsilon < 1/2$ then $d(\tilde{f}, \tilde{g}) < 2\epsilon$. This is because $(\tilde{f} - \tilde{g}) \in \mathcal{C}([0,1], \mathbf{R})$ vanishes at 0 and only takes on values within ϵ of an integer, so cannot escape from $N_\epsilon(0)$.

Now, a continuous function from $\mathbf{R}/\mathbf{Z}$ to $\mathbf{R}/\mathbf{Z}$ is just a function $f \in \mathcal{C}([0,1], \mathbf{R}/\mathbf{Z})$ such that $f(0) = f(1)$. For such a function, $\tilde{f}(1) \in \mathbf{Z}$. This integer $\tilde{f}(1)$ is the *winding number* (a.k.a. "degree") of f. For instance, for any integer n the map $a \mapsto na$ (why is this well-defined?) has winding number n. From our analysis thus far it follows that if $d(f,g) < 1/2$ then f, g have the same winding numbers. Thus if $\phi, \psi : \mathbf{R}/\mathbf{Z} \to \mathbf{C}^*$ satisfy $|\phi(a)| > |\phi(a) - \psi(a)|$ for all $a \in \mathbf{R}/\mathbf{Z}$ then $\alpha \circ \phi$ and $\alpha \circ \psi$ have the same winding number.

By continuity of P and Heine-Borel in $\mathbf{C}$ (or a more explicit ϵ, δ computation), the map $[0, \infty) \to \mathcal{C}(\mathbf{R}/\mathbf{Z}, \mathbf{R}/\mathbf{Z})$ taking r to the function $f_r : a \mapsto \alpha(P(re(a)))$ is continuous. Thus the winding number of f_r is independent of r. But for $r = 0$ the function f_r is constant, so has winding number zero. For $r > \max(1, \sum_{k=0}^{n-1} |a_k|/|a_n|)$, we have

$$|P(re(a)) - a_n(re(a))^n| < |a_n(re(a))^n| \; \left[= |a_n| r^n\right],$$

so f_r has the same winding number as $a \mapsto a_n r^n e(an)$ which is n. This completes the desired contradiction, and thus the proof of the Fundamental Theorem of Algebra.