Math 272y: Rational Lattices and their Theta Functions

16 September 2019: Introduction to theta functions of lattices

Recall that for us the *norm* of a vector $\mathbf{x}$ in a lattice or inner-product space is $\langle \mathbf{x}, \mathbf{x} \rangle$ (and not the length $\langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$). Fix some inner-product space V of dimension n. For any lattice $L \subset V$ and each $k \in \mathbf{R}$, define

$$N_k(L) = \#\{\mathbf{x} \in L \mid \langle \mathbf{x}, \mathbf{x} \rangle = k\}. \tag{1}$$

The set is finite; indeed for any $k_0 \in \mathbf{R}$ the sum

$$\sum_{k \le k_0} N_k(L) = \#\{\mathbf{x} \in L \mid \langle \mathbf{x}, \mathbf{x} \rangle \le k_0\} \tag{2}$$

is finite because L is discrete and the subset $\{\langle \mathbf{x}, \mathbf{x} \rangle \le k_0\}$ in V is compact (a closed ball if $k_0 \ge 0$, empty if $k_0 < 0$). The $N_k(L)$ are important invariants of L. For example, in the context of sphere packing one is interested in the *minimal (nonzero) norm* of L, which is the smallest $k > 0$ for which $N_k(L) > 0$, and the *kissing number* of L, which is the value of $N_k(L)$ for that k.[1] The *theta function* or *theta series* Θ_L is a generating function that encodes these invariants $N_k(L)$:

$$\Theta_L(q) := \sum_{\mathbf{x} \in L} q^{\langle \mathbf{x}, \mathbf{x} \rangle / 2} = 1 + \sum_{k > 0} N_{2k}(L) q^k. \tag{3}$$

Note that $N_0(L) = 1$; the factors of 2 are convenient as we soon see. The sum in (3) converges absolutely if $0 \le q < 1$, because the sum (2) is $O_L(k_0^{n/2})$ as $k_0 \to \infty$ (indeed it is asymptotic to $c_L k_0^{n/2}$, where c_L is $\operatorname{disc}(L)^{-1/2}$ times the volume $\pi^{n/2}/\Gamma((n/2)+1)$ of a unit sphere in V). This makes it easy to justify the following derivation of the product formula for the theta function of a direct sum:

$$\begin{aligned} \Theta_{L_1 \oplus L_2}(q) &= \sum_{(\mathbf{x}_1, \mathbf{x}_2) \in L_1 \oplus L_2} q^{\langle \mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_1 + \mathbf{x}_2 \rangle / 2} \\ &= \sum_{(\mathbf{x}_1, \mathbf{x}_2) \in L_1 \oplus L_2} q^{(\langle \mathbf{x}_1, \mathbf{x}_1 \rangle + \langle \mathbf{x}_2, \mathbf{x}_2 \rangle)/2} \\ &= \sum_{\mathbf{x}_1 \in L_1} q^{\langle \mathbf{x}_1, \mathbf{x}_1 \rangle / 2} \sum_{\mathbf{x}_2 \in L_2} q^{\langle \mathbf{x}_2, \mathbf{x}_2 \rangle / 2} \\ &= \Theta_{L_1}(q)\, \Theta_{L_2}(q). \end{aligned} \tag{4}$$

[1] The minimal norm is $(2r)^2$ where r is the radius of the largest ball in V whose translates by L have disjoint interiors. These translates constitute the sphere packing associated with L. Each of them is tangent to κ others, where κ is the kissing number.

For example,

$$\Theta_{\mathbf{Z}^n}(q) = \Theta_{\mathbf{Z}}(q)^n = \left(\sum_{m=-\infty}^{\infty} q^{m^2/2} \right)^n . \tag{5}$$

Even clearer is the identity relating the theta functions of L and $L\langle c\rangle$ for any $c > 0$:

$$\Theta_{L\langle c\rangle}(q) = \Theta_L(q^c). \tag{6}$$

A more remarkable identity relates the theta functions of a lattice and its dual:

Proposition (functional equation for theta series). *For any lattice L in V we have*

$$\Theta_{L^*}(e^{-2\pi t}) = \operatorname{disc}(L)^{1/2} t^{-n/2} \Theta_L(e^{-2\pi/t}). \tag{7}$$

for all $t > 0$.

This is a special case of the *Poisson summation formula* in V, which relates the sums of a Schwartz function over L and of its Fourier transform over L^*. Here a *Schwartz function* is a C^∞ function $f : V \to \mathbf{C}$ such that f and all its partial derivatives decay as $o(\langle \mathbf{x}, \mathbf{x}\rangle^k)$ for all k as $\langle \mathbf{x}, \mathbf{x}\rangle \to \infty$. The *Fourier transform* $\hat{f} : V^* \to \mathbf{C}$ is defined by[2]

$$\hat{f}(\mathbf{y}) = \int_{\mathbf{x}\in V} f(\mathbf{x})\, e^{2\pi i\langle \mathbf{x},\mathbf{y}\rangle}\, d\mu(\mathbf{x}), \tag{8}$$

and is a Schwartz function if f is. This requires a choice of invariant measure μ. Our V will always be equipped with an inner product, which identifies V with V^* and gives a natural choice of μ.

Theorem (Poisson summation in $\mathbf{R}^n$). *Let L be any lattice in a finite-dimensional real inner product space V. Then*

$$\sum_{\mathbf{x}\in L} f(\mathbf{x}) = (\operatorname{disc} L)^{-1/2} \sum_{\mathbf{y}\in L^*} \hat{f}(\mathbf{y}) \tag{9}$$

for all Schwartz functions $f : V \to \mathbf{C}$.[3]

[2]It is important that we have the factor $2\pi i$ in the exponent. It is not so important for us whether the exponent is $2\pi i\langle \mathbf{x}, \mathbf{y}\rangle$ or $-2\pi i\langle \mathbf{x}, \mathbf{y}\rangle$ (the latter is used in many sources), because our formulas are centrally symmetric.

[3]The Schwartz condition is much more restrictive than is needed to justify Poisson summation (though Schwartz functions suffice for all our applications to lattices and theta series). For example, Poisson summation holds if there exists $\delta > 0$ for which both $f(x)$ and $\hat{f}(x)$ are $O(\langle \mathbf{x}, \mathbf{x}\rangle^{-(n/2)-\delta})$ as $\langle \mathbf{x}, \mathbf{x}\rangle \to \infty$ (see e.g. Corollary 2.6 in Chapter VII of Stein and Weiss's *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press 1971). One application: our derivation of (9) is valid also for $n = 1$ and $f(x) = e^{-|x|}$, $\hat{f}(y) = 2/((2\pi y)^2 + 1)$; this yields a closed form

$$\frac{1}{2}\left(\frac{\pi}{c} \frac{e^{2\pi c} + 1}{e^{2\pi c} - 1} - \frac{1}{c^2} \right)$$

for $\sum_{n=1}^{\infty} 1/(n^2 + c^2)$ (any $c > 0$), whose power-series expansion about $c = 0$ then lets us recover the value of $\zeta(k)$ as a rational multiple of π^k for all even $k > 0$.

Note that the $\mathbf{y} = 0$ term in (9) is

$$(\operatorname{disc} L)^{-1/2} \hat{f}(0) = \frac{1}{\operatorname{Vol}(V/L)} \int_{\mathbf{x}\in V} f(\mathbf{x})\, d\mu(\mathbf{x}). \tag{10}$$

After multiplying (9) by $\operatorname{disc}(L)^{1/2}$, we can thus interpret the left-hand side as a Riemann sum approximating the integral $\hat{f}(0)$. The terms $\hat{f}(\mathbf{y})$ for nonzero $\mathbf{y} \in L^*$ then measure the discrepancy between this Riemann sum and the integral.

Proof of Poisson summation: Define $F : V \to \mathbf{C}$ by

$$F(\mathbf{z}) = \sum_{\mathbf{x}\in L} f(\mathbf{x} + \mathbf{z}). \tag{11}$$

Because f is Schwartz, the sum converges absolutely to a C^∞ function, whose value at $\mathbf{z} = 0$ is the left-hand side of (9) Since $F(\mathbf{z}) = F(\mathbf{x} + \mathbf{z})$ for all $\mathbf{z} \in V$ and $\mathbf{x} \in L$, the function descends to a C^∞ function on V/L, and thus has a Fourier expansion

$$F(\mathbf{z}) = \sum_{\mathbf{y}\in L^*} \hat{F}(-\mathbf{y})\, e^{2\pi i\langle \mathbf{y},\mathbf{z}\rangle}, \tag{12}$$

where

$$\hat{F}(\mathbf{y}) = \frac{1}{\operatorname{Vol}(V/L)} \int_{\mathbf{z}\in V/L} F(\mathbf{z})\, e^{2\pi i\langle \mathbf{z},\mathbf{y}\rangle}\, d\mu(\mathbf{z}). \tag{13}$$

Note that the vectors $\mathbf{y} \in L^*$ are exactly those for which $e^{2\pi i\langle \mathbf{z},\mathbf{y}\rangle}$ is well-defined on V/L. Now choose a fundamental domain R for V/L; for instance, let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be generators of L and set $R = \{a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n : 0 \le a_i < 1\}$. Then we have

$$\begin{aligned}
\operatorname{Vol}(V/L)\hat{F}(\mathbf{y}) &= \int_{\mathbf{z}\in R} F(\mathbf{z})\, e^{2\pi i\langle \mathbf{y},\mathbf{z}\rangle}\, d\mu(\mathbf{z}) \\
&= \int_{z\in R} \sum_{\mathbf{x}\in L} f(\mathbf{x}+\mathbf{z})\, e^{2\pi i\langle \mathbf{y},\mathbf{z}\rangle}\, d\mu(\mathbf{z}) \\
&= \sum_{\mathbf{x}\in L} \int_{\mathbf{z}\in R-\mathbf{x}} f(\mathbf{z})\, e^{2\pi i\langle \mathbf{y},\mathbf{z}\rangle}\, d\mu(\mathbf{z}) \\
&= \int_{\mathbf{z}\in V} f(\mathbf{z})\, e^{2\pi i\langle \mathbf{y},\mathbf{z}\rangle}\, d\mu(\mathbf{z}) \;=\; \hat{f}(\mathbf{y}),
\end{aligned} \tag{14}$$

where we used in the last step the fact that V is the disjoint union of the translates $R - \mathbf{x}$ of R by lattice vectors. Thus (12) becomes

$$F(\mathbf{z}) = \frac{1}{\operatorname{Vol}(V/L)} \sum_{\mathbf{y}\in L^*} \hat{f}(-\mathbf{y})\, e^{2\pi i\langle \mathbf{y},\mathbf{z}\rangle}. \tag{15}$$

Taking $\mathbf{z} = 0$ we obtain (9), Q.E.D.

The functional equation (7) is then the special case $f(\mathbf{x}) = \exp(-\pi\langle\mathbf{x},\mathbf{x}\rangle/t)$ of (9).

Proof of the functional equation (7) for theta series: Let

$$f(\mathbf{x}) = \exp(-\pi\langle\mathbf{x},\mathbf{x}\rangle/t) \tag{16}$$

in (9). We claim that

$$\hat{f}(\mathbf{y}) = t^{n/2}\exp(-\pi\langle\mathbf{y},\mathbf{y}\rangle t). \tag{17}$$

Choosing any orthogonal coordinates $(x_1, \ldots, x_n)$ for V, we see that the integral (8) defining $\hat{f}(\mathbf{y})$ factors as

$$\prod_{j=1}^{n}\int_{-\infty}^{\infty} e^{-\pi x_j^2/t} e^{2\pi i x_j y_j}\, dx_j,$$

which reduces our claim to the case $n = 1$, which is the familiar definite integral

$$\int_{-\infty}^{\infty} e^{-\pi x^2/t}\, e^{2\pi i x y}\, dx = t^{1/2} e^{-\pi t y^2}.$$

Using these f and $\hat{f}$ in the Poisson summation formula (9) we deduce the functional equation (7), Q.E.D.

Already the first example, with $n = 1$ and $L = L^* = \mathbf{Z}$, is surprising and important:

$$\sum_{m=-\infty}^{\infty} e^{-\pi m^2 t} = t^{-1/2}\sum_{m=-\infty}^{\infty} e^{-\pi m^2/t}. \tag{18}$$

A famous application is Riemann's proof of the analytic continuation and functional equation of the zeta function $\zeta(s) = \sum_{m=1}^{\infty} m^{-s}$: multiply $\Theta_{\mathbf{Z}}(e^{-2\pi t}) - 1$ by $t^{s/2}\, dt/t$ and integrate termwise over $0 < t < \infty$ to find

$$\begin{aligned}
\int_0^{\infty}(\Theta_{\mathbf{Z}}(e^{-2\pi t}) - 1)\, t^{s/2}\frac{dt}{t} &= 2\sum_{m=1}^{\infty}\int_0^{\infty} e^{-\pi m^2 t}\, t^{s/2}\frac{dt}{t} \\
&= 2\sum_{m=1}^{\infty}(\pi m^2)^{-s/2}\Gamma(s/2) \\
&= 2\pi^{-s/2}\Gamma(s/2)\zeta(s) = \xi(s)
\end{aligned} \tag{19}$$

for $\mathrm{Re}(s) > 0$; then split the integral as $\int_0^{\infty} = \int_0^1 + \int_1^{\infty}$ and apply (18) to $\int_0^1$ to obtain

$$\xi(s) + \frac{1}{s} + \frac{1}{1-s} = \frac{1}{2}\int_1^{\infty}(\Theta_{\mathbf{Z}}(e^{-2\pi t}) - 1)(t^{s/2} + t^{(1-s)/2})\frac{dt}{t} \tag{20}$$

for $0 < \mathrm{Re}(s) < 1$. To recover the analytic continuation of $\xi(s)$ to all of $\mathbf{C}$ (with simple poles at $s = 0$ and $s = 1$), observe that the integral in (20) is an analytic function of s on all of $\mathbf{C}$, because $\Theta_{\mathbf{Z}}(e^{-2\pi t}) - 1$ decays exponentially as $t \to \infty$; the functional equation $\xi(s) = \xi(1-s)$ then follows from the symmetry of the integral under $s \leftrightarrow 1 - s$.

Applying the same definite integral to Θ_L for a general lattice L in V, but using t^s instead of $t^{s/2}$, we obtain

$$\int_0^\infty (\Theta_L(e^{-2\pi t}) - 1)\, t^s \frac{dt}{t} = \pi^{-s}\Gamma(s)\zeta_L(s) = \xi_L(s), \tag{21}$$

where ζ_L is the zeta function of L, defined by

$$\zeta_L(s) := \sum_{\substack{v \in L \\ v \neq 0}} \langle \mathbf{x}, \mathbf{x} \rangle^{-s}, \tag{22}$$

and ξ_L is defined by the last equality in (21). Transforming the integral as before, and using the functional equation (7) relating Θ_L with Θ_{L^*}, we obtain the identity

$$\xi_{L^*}\left(\frac{n}{2} - s\right) = \mathrm{disc}(L)^{1/2} \xi_L(s), \tag{23}$$

generalizing the functional equation for the Riemann zeta function.

For a rather more frivolous application of (18) (and one admittedly unrelated to our main topic), differentiate both sides of (18) with respect to t and set $t = 1$ to find[4]

$$\frac{d}{dt}\Theta_{\mathbf{Z}}(e^{-2\pi t})\bigg|_{t=1} = -\frac{1}{4}\Theta_{\mathbf{Z}}(e^{-2\pi}),$$

whence

$$1 + 2\sum_{m=1}^{\infty} e^{-\pi m^2 t} = \Theta_{\mathbf{Z}}(e^{-2\pi}) = 8\pi \sum_{m=1}^{\infty} m^2 e^{-\pi m^2 t}.$$

Therefore $8\pi < e^\pi + 2$, with a rather small difference

$$e^\pi + 2 - 8\pi \approx (32\pi - 2)e^{-3\pi} = 0.00795+ \tag{24}$$

because $e^{-3\pi}$ is tiny. Subtracting from this the error $22 - 7\pi = 0.00885+$ in the familiar approximation $\pi \approx 22/7$ yields the striking

$$e^\pi - \pi = 19.999099979+,$$

[4]The next identity was surely known to Riemann, if not to Poisson himself: an equivalent form appears in the last displayed equation before formula (2) in Edwards' *Riemann's Zeta Function* (New York: Academic Press, 1974), page 17, where Edwards recites Riemann's derivation of a series representation for $\xi(\frac{1}{2} + it)$.

which has even featured in an xkcd comic (#217).

We next consider Θ_L as a function of a complex variable. For general lattices L we cannot make sense of $\Theta_L(q)$ as a function of q in a neighborhood of $q = 0$ in $\mathbf{C}$, because the exponents $\langle \mathbf{v}, \mathbf{v} \rangle/2$ in (3) need not be integers. However, the change of variables $q = e^{-2\pi t}$ suggested by the functional equation (7) yields a function of t that extends to a holomorphic function on the half-plane $\mathrm{Re}(t) > 0$. That functional equation then extends to this half-plane, either by analytic continuation or by using the same proof.

Suppose now that L is self-dual. Then $\Theta_L = \Theta_{L^*}$, so the functional equation (3) relates the values at t and $1/t$ of the same function. Also, each exponent $\langle \mathbf{v}, \mathbf{v} \rangle/2$ is in $\frac{1}{2}\mathbf{Z}$ because L is integral. Thus $\Theta_L(e^{-2\pi t})$ is also invariant under the imaginary translation $t \mapsto t + 2i$. Combining this invariance with the functional equation we then obtain further identities, one for each fractional linear transformation generated by $t \mapsto 1/t$ and $t \mapsto t + 2i$. If furthermore L is even then the exponents $\langle \mathbf{v}, \mathbf{v} \rangle/2$ are all integral, so $\Theta_L(e^{-2\pi t})$ is invariant not just under $t \mapsto t + 2i$ but already under $t \mapsto t + i$.

We make the coefficients of these transformations integral using the further change of variable $t = iz$. Then z is in the Poincaré upper half-plane

$$\mathcal{H} = \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\},$$

and

$$q = e^{2\pi i z}. \tag{25}$$

Our transformations $t \mapsto 1/t$, $t \mapsto t + 2i$, and $t \mapsto t + i$ then become

$$S : z \mapsto -1/z, \qquad T^2 : z \mapsto z + 2, \qquad T : z \mapsto z + 1,$$

acting on $\mathcal{H}$.

We are thus led to define for any lattice L in V the function

$$\theta_L(z) := \Theta_L(e^{2\pi i z}) = 1 + \sum_{\substack{k>0 \\ N_k(L) \neq 0}} N_k(L) e^{\pi i k z}, \tag{26}$$

on $\mathcal{H}$. If L is self-dual then θ_L satisfies the functional equations

$$\theta_L(z + 2) = \theta_L(z), \quad \theta_L(-1/z) = (z/i)^{n/2} \theta_L(z), \tag{27}$$

where "$(z/i)^{n/2}$" is the nth power of the principal square root of z/i (that is, the square root with positive real part); if moreover L is even then the first identity in (27) can be replaced by $\theta_L(z + 1) = \theta_L(z)$.

We shall soon see that this makes θ_L a modular form of weight $n/2$, and give some applications of its modularity; for example, we shall give an easy proof that all even unimodular lattices of rank 8 are isomorphic, and count the automorphisms. We shall then go back to the geometry of Euclidean lattices to develop the theory of root lattices (lattices generated by their vectors of norm 2), which will let us use the modularity of θ_L to give several further classification results. We shall then generalize to weighted theta functions of self-dual lattices, and then to the modularity of theta functions of rational positive-definite lattices; this will greatly expand our ability to use modular forms to study positive-definite quadratic forms.