Math 272y: Rational Lattices and their Theta Functions

4 and 6 November 2019:
More about Λ_{24}; extremal theta functions and lattices, and spherical designs

Venkov's proof of the Niemeier classification started with the fact that if N is a Niemeier lattice and P is any harmonic polynomial of degree 2 then $\theta_{N,P}$ is a modular cusp form of weight 14, and is therefore zero — whence the roots of N constitute a spherical 2-design. In the special case where N is the Leech lattice, if P is a nonconstant harmonic polynomial of degree d then $\theta_{N,P}$ is a modular form of weight $12 + d$ that vanishes to order at least 2 at the cusp, and is thus zero if $d < 12$, and also if $d = 14$. Hence each shell of the Leech lattice is a spherical 11-design, and indeed an "$11\frac{1}{2}$-design" as Venkov calls it. (In general, for odd d Venkov says a centrally symmetric finite subset S of a sphere is a "$d\frac{1}{2}$-design" if $\sum_{x\in S} P(x) = 0$ for all nonconstant spherical polynomials P with $\deg P \leq d$ or $\deg P = d + 3$.) This property uses only the assumption that N is an even self-dual lattice of rank 24 with no vectors of norm 2, not the fact that Λ_{24} is the unique such lattice. As with the $7\frac{1}{2}$-design property for E_8, we can thus regard this $11\frac{1}{2}$-design property either as a strong constraint on any such lattice N or as a tool for obtaining detailed information about the configuration of short vectors in Λ_{24}.

We illustrate this by tabulating some statistics for the distribution of the 196560 vectors in the first shell of Λ_{24}. We then generalize to self-dual even lattices in $\mathbf{R}^n$ with "extremal" theta series for other $n \equiv 0 \bmod 8$, and conclude this chapter with statistics for $n = 32, 48, 72$.

As we did for E_8, we fix some nonzero $v_0 \in \Lambda_{24}$, and let N_k ($k \in \mathbf{Z}$) be the number of $v \in \Lambda_{24}$ with $\langle v, v\rangle = 4$ and $\langle v_0, v\rangle = k$. Then $N_{-k} = N_k$ for all k, and $N_k = 0$ if $k^2 > 4\langle v_0, v_0\rangle$ by Cauchy-Schwarz. Here the fact that Λ_{24} has no roots gives us a further condition: $N_k = 0$ if $|k| > \frac{1}{2}\langle v_0, v_0\rangle$, except if $\langle v_0, v_0\rangle = 4$ when $N_{\pm 4} = 1$. Indeed if $\langle v_0, v\rangle > \frac{1}{2}\langle v_0, v_0\rangle$ then

$$\langle v - v_0, v - v_0\rangle = \langle v, v\rangle - 2\langle v_0, v\rangle + \langle v_0, v_0\rangle < \langle v, v\rangle,$$

but $\langle v, v\rangle = 4$ so $v - v_0 = 0$; likewise if $\langle v_0, v\rangle < -\frac{1}{2}\langle v_0, v_0\rangle$ then $v + v_0 = 0$. This together with the 11-design property gives us enough conditions to solve for the N_k as long as $\langle v_0, v_0\rangle \leq 10$. For each of the possible norms $\langle v_0, v_0\rangle = 4, 6, 8, 10$, we list the number of such v_0 (these counts are the q^2, q^3, q^4, q^5 coefficients of $\theta_{\Lambda_{24}} = \mathsf{E}_4^3 - 720\Delta$), followed by the values of N_k for $|k| \leq 5$.

$\langle v_0, v_0\rangle$	#	N_0	$N_{\pm1}$	$N_{\pm2}$	$N_{\pm3}$	$N_{\pm4}$	$N_{\pm5}$
4	$196560 = 2^4 3^3 5{\cdot}7{\cdot}13$	93150	47104	4600	0	1	0
6	$16773120 = 2^{12} 3^2 5{\cdot}7{\cdot}13$	75900	48600	11178	552	0	0
8	$398034000 = 2^4 3^7 5^3 7{\cdot}13$	65780	47104	16192	2048	46	0
10	$4629381120 = 2^{14} 3^3 5{\cdot}7{\cdot}13{\cdot}23$	58806	45100	19450	4050	275	2

As was true for E_8, the fact that each of these $N_{\pm k}$ is the same for all v_0 of the same norm also reflects the fact that $\mathrm{Aut}(\Lambda_{24})$ (a.k.a. the Conway group $Co_0 = 2.Co_1$) acts transitively on each of the first four shells of lattice vectors. (See page 181 of the ATLAS [second page of the Co_1 entry] for a list of orbit sizes and stabilizers for the action of Co_0 on nonzero v_0 with $\langle v_0, v_0 \rangle \leq 32$.) But the common counts $N_{\pm k} = N_{\pm k}(v_0)$ still carry information about the structure of Λ_{24}.

For $\langle v_0, v_0 \rangle = 4$, the count $N_0 = 93150$ is the kissing number of the orthogonal complement of v_0 in Λ_{24}; this complement is the "laminated lattice" Λ_{23}, which is the unique even lattice of rank 23 and discriminant 4 with no roots. This is conjectured to be the maximal kissing number in $\mathbf{R}^{23}$, even without the lattice condition. The count $N_2 = 4600$ is what is sometimes called the "necking number" of Λ_{24}, that is, the number of lattice vectors at minimal distance from two closest lattice vectors, here 0 and v_0. (This number is well-defined at least if the automorphism group acts transitively on minimal vectors, as is the case here.) It is also the kissing number of the "shorter Leech lattice" O_{23}, which is the unique lattice strictly between Λ_{23} and Λ_{23}^*, and the unique self-dual lattice of rank at most 23 without vectors of norm 1 or 2 (this too can be deduced from the Niemeier classification via gluing); the minimal norm is 3, attained by the vectors $v - \frac{1}{2}v_0$ with v varying over the 4600 "necking" vectors. The stabilizer of v_0 is Conway's sporadic group Co_2, which is also the determinant $+1$ subgroup of $\mathrm{Aut}(O_{23})$, and acts transitively on the 2300 pairs of minimal vectors of O_{23}. As for $N_{\pm 1} = 47104$, that is the number of minimal vectors in each non-integral coset of Λ_{23} in Λ_{23}^*; we shall use this number again when we reach the counts for $\langle v_0, v_0 \rangle = 10$.

For $\langle v_0, v_0 \rangle = 6$, the count $N_0 = 75900$ is the kissing number of the orthogonal complement of v_0 in Λ_{24}, which this time is unique even lattice of rank 23 and discriminant 6 with no roots. The other $N_{\pm k}$ again count minimal vectors in cosets of that lattice in its dual. Here the stabilizer of v_0, and the group of determinant $+1$ automorphisms of the orthogonal complement, is Conway's sporadic group Co_2. As usual, the minimal vectors v with $\langle v_0, v \rangle = \frac{1}{2}\langle v_0, v_0 \rangle$ come in pairs $\{v, v_0 - v\}$; here there are $N_3/2 = 276$ pairs, and it is known that Co_3 acts *doubly* transitively on them. For the point stabilizer, see the discussion of $\langle v_0, v_0 \rangle = 10$.

We have already encountered $\langle v_0, v_0 \rangle = 8$. The count $N_4 = 46$ gives us another entry point into Conway's characterization of Λ_{24}: the orthogonal frame of norm-8 vectors in $v_0 + 2L$ consists of the 46 vectors $2v - v_0$, together with v_0 itself and $-v_0$. Recall that we constructed Λ_{24} as a 2-neighbor of the A_1^{24} Niemeier lattice $\{x \in \mathbf{Z}^{24}\langle 1/2 \rangle : x \bmod 2 \in \mathcal{G}_{24}\}$. In those coordinates, we may take $v_0 = 4e_1$; then the 46 vectors $v \in \Lambda_{24}$ with $\langle v, v \rangle = \langle v_0, v \rangle = 4$ are $e_1 \pm e_j$ $(2 \leq j \leq 24)$, and the 2^{11} short vectors with $\langle v_0, v \rangle = 3$ are $\frac{3}{2}e_1 + \frac{1}{2}\sum_{j=2}^{24} \epsilon_j e_j$ for certain $\epsilon_j = \pm 1$ determined by the Golay code $\mathcal{G}_{24}$.

The situation for $\langle v_0, v_0 \rangle = 10$ recalls our observation for vectors of norm 6 in E_8: in each case, the last nonzero N_k is 2, and we deduce that v is uniquely a sum of two minimal vectors. Here we find that *every nonzero Leech vector of norm* 10 *is uniquely the sum of two minimal vectors*, necessarily with inner product 1. As a consistency check, the number of such unordered pairs $\{v_1, v_2\}$ of minimal vectors is $\frac{1}{2}196560 N_1(v_1) = 196560 \cdot 47104/2$, which indeed agrees with the count 4629381120 of norm-10 vectors in Λ_{24}. Alternatively we can use the observation that $\frac{1}{2}N_4(\Lambda_{24})N_1(v_1) = N_{10}(\Lambda_{24})$ to prove that every v_0 of norm 10 has such a representation $v_1 + v_2$: if not, then some v_0 would have two different representations, but this is not possible once $\langle v_0, v_0 \rangle > 8$ because if $v_0 = v_1 + v_2 = v_1' + v_2'$

then

$$4 \le \langle v_1 + v_1' - v_0, v_1 + v_1' - v_0 \rangle = 8 + 2\langle v_1, v_1' \rangle - \langle v_0, v_0 \rangle < 2\langle v_1, v_1' \rangle,$$

whence $\langle v_1, v_1' \rangle > 2$, which we already know is not possible. Thus the stabilizer of v_0 in $\mathrm{Aut}(\Lambda_{24})$ is the stabilizer of the plane spanned by v_1, v_2. The pointwise stabilizer of v_1 and v_2 is yet another sporadic simple group McL, and the stabilizer of $\{v_1, v_2\}$ is $\mathrm{Aut}(\mathrm{McL}) = \mathrm{McL} : 2$. This group was already discovered a few years earlier (1969) by J. McLaughlin, who constructed $\mathrm{McL}:2$ as the automorphism group of a strongly regular graph on 275 vertices. In Λ_{24}, these vertices arise as the $N_4(v_0) = 275$ short vectors v with $\langle v_0, v \rangle = 4$, and their graph adjacency is determined by their inner products in Λ_{24}. Since $v_1 - v_2$ has norm 6, this group is also contained in Co_3, and indeed is the point stabilizer in the action of Co_3 on the 276 pairs of short vectors that sum to $v_1 - v_2$ (which split $1 + 275 + 275 + 1$ according to their inner product with v_0).

We take this one step further. The action of $\mathrm{Aut}(\Lambda_{24})$ on the norm-12 shell cannot be transitive, because $N_{12}(\Lambda_{24}) = 34417656000 = 2^6 3^3 5^3 7 \cdot 13 \cdot 17 \cdot 103$ but $\mathrm{Aut}(\Lambda_{24})$ cannot contain a 103-cycle. (In fact

$$|\mathrm{Aut}(\Lambda_{24})| = 8315553613086720000 = 2^{22} 3^9 5^4 7^2 11 \cdot 13 \cdot 23,$$

as Conway showed in his characterization of the Leech lattice.) There is an easy invariant, though: some but not all vectors v_0 of norm 12 are the sums of two minimal vectors. Since such a representation, if it exists, is unique, there are

$$\frac{1}{2} 196560 N_2(v_1) = 196560 \cdot 4600/2 = 452088000$$

choices of v_0 with $\langle v_0, v_0 \rangle = 12$ and $N_6(v_0) = 2$, leaving $33965568000 = 2^{12} 3^6 5^3 7 \cdot 13$ for which $\langle v_0, v_0 \rangle = 12$ and $N_6(v_0) = 0$. Each of these subcounts is a factor of $|\mathrm{Aut}(\Lambda_{24})|$, and indeed it turns out that $\mathrm{Aut}(\Lambda_{24})$ acts transitively on each of the subsets $N_6(v_0) = 2$ and $N_6(v_0) = 0$ of the norm-12 shell. In particular, in each subset the other counts $N_k(v_0)$ with $|k| \le 5$ must be constant. We can calculate them using the 11-design property, finding

$$(N_0, N_{\pm 1}, N_{\pm 2}, N_{\pm 3}, N_{\pm 4}, N_{\pm 5}) = \begin{cases} (55530, 41472, 22518, 5632, 891, 0), & \text{if } N_6 = 2; \\ (53682, 43056, 21528, 6072, 759, 24), & \text{if } N_6 = 0. \end{cases} \quad (1)$$

(If we did not know *a priori* that $N_6(v_0) \le 2$, we could deduce this fact from the 11-design calculation: $N_6(v_0)$ must be even, and if $N_6(v_0) > 2$ then $N_5(v_0) = 24 - 12N_6 < 0$.)

In the former case, v_0 is contained in one of the $452088000/6 = 75348000$ sublattices of Λ_{24} isomorphic with $A_2\langle 2 \rangle$. Here the stabilizer is not sporadic: v_0 has stabilizer isomorphic with $\mathrm{PSU}_6(\mathbf{Z}/2\mathbf{Z})$, and the stabilizer of $A_2\langle 2 \rangle$ is $\mathrm{PSU}_6(\mathbf{Z}/2\mathbf{Z}) : S_3$. The 891 vectors counted by $N_4(v_0)$ correspond to the maximal isotropic spaces for a unitary form such as $\sum_{j=1}^6 x_j^3$ over $\mathbf{F}_4$; this is reminiscent of the appearance of the configuration of 27 lines on a cubic surface in E_6, which is the orthogonal complement of A_2 in E_8.

In the latter case (when $N_6 = 0$ in (1)), we get 24 vectors v whose projections $v - \frac{5}{12} v_0$ have norm $4 - (5^2/12) = 23/12$ and pairwise inner products $-1/12$. Hence these projections form a regular simplex, and the original 24 vectors generate a finite-index subgroup of Λ_{24}. In fact we can use this to start another proof of the characterization of Λ_{24}; but it soon merges with Conway's proof (as one might guess from $N_4(v_0) = 759$, which is the number of minimal-weight words in $\mathcal{G}_{24}$): v_0 is the vector $\sum_{j=1}^{24} e_j$, and the 24 vectors v are $2e_j - \frac{1}{2}v_0$.

Now to generalize the $7\frac{1}{2}$- and $11\frac{1}{2}$-design properties of E_8 and Λ_{24}.

Lemma. *Suppose L is an even self-dual lattice in $\mathbf{R}^n$, and let $\delta = \delta(L) = 6N_{\min}(L) - (n/2)$ where $N_{\min}(L)$ is the minimal nonzero norm. If $\delta(L) = 0$ then every shell of L is a spherical 3-design. If $\delta(L) > 0$ then every shell of L is a spherical $d\frac{1}{2}$-design where $d = \delta - 1$.*

Proof. Let $\nu = N_{\min}(L)$, which is some positive even integer. For $\delta = 0$ we are to prove that $\theta_{N,P} = 0$ for every harmonic polynomial P of degree 2. Here $n = 12\nu$, so $\theta_{N,P}$ is a modular form of weight $6\nu + 2$ that vanishes to order at least $\nu/2$ at the cusp, and is thus $\Delta^{\nu/2}$ times a modular form of weight 2. The only such form is zero, so $\theta_{N,P} = 0$. For $\delta > 0$ we are to prove that $\theta_{N,P} = 0$ for every nonconstant harmonic polynomial P of degree at most d, and also for $\deg(P) = d + 3$. Here $n = 12\nu - 2\delta$, so $\theta_{N,P}$ is a modular form of weight $6\nu - \delta + \deg(P)$ that again vanishes to order at least $\nu/2$ at the cusp, and is thus $\Delta^{\nu/2}$ times a modular form of weight $\deg(P) - \delta$. If $\deg(P) = \delta + 2$ then this weight is 2, while if $\deg(P) \leq \delta - 1$ then the weight is negative; in either case we deduce that $\theta_{N,P} = 0$. $\square$

So, how large can $\delta(L)$ get? The dimension of the space of modular forms of weight $n/2$ suggests that $N_{\min}(L)$ might get as large as $2\lfloor n/24 \rfloor + 2$, as it does for $n = 8, 16, 24$, but no larger: θ_L must be $\mathsf{E}_4^{n/8} + \sum_{j=1}^{\lfloor n/24 \rfloor} a_j \Delta^j \mathsf{E}_4^{(n/8)-3j}$ for some a_j, and if $N_{\min}(L) > 2\lfloor n/24 \rfloor + 2$ then $a_1, a_2, \ldots, a_{\lfloor n/24 \rfloor}$ are successively determined by the condition that the $q, q^2, \ldots, q^{\lfloor n/24 \rfloor}$ coefficients of θ_L vanish, at which point θ_L is completely determined. This is what happened for $n = 8, 16$ (with no a_j) and $n = 24$ (with $a_1 = -720$). In general, we call the resulting modular form the *extremal theta function* of weight $n/2$ (whether or not it is actually the theta function of some lattice); it is the unique modular form θ of this weight such that $\theta - 1$ vanishes to order greater than $\lfloor n/24 \rfloor$ at the cusp $q = 0$. Conceivably, though, this form's $q^{\lfloor n/24 \rfloor + 1}$ coefficient might be negative or zero; in the former case, $N_{\min}(L)$ could not be as large as $2\lfloor n/24 \rfloor + 2$ for any even self-dual lattice of rank n, and in the latter case, $N_{\min}(L)$ could be yet larger. Siegel[1] showed that in fact the $q^{\lfloor n/24 \rfloor + 1}$ coefficient is always positive, and the result was later generalized to other kinds of extremal theta functions (and extremal weight enumerators for self-dual codes). The general setting is as follows.

Let $f(q)$ and $g(q)$ be power series of the form

$$f(q) = 1 + O(q), \quad g(q) = q + O(q^2), \tag{2}$$

and fix some r with $0 \leq r \leq 1$. (In our setting, $(f, g) = (\mathsf{E}_4^3, \Delta)$, and $r = 0$, $\frac{1}{3}$, or $\frac{2}{3}$ according as n is 0, 8, or $16 \bmod 24$.) For a nonnegative integer k and any $a_0, a_1, \ldots, a_k$ there exists a

[1] Carl Ludwig Siegel: Berechnung von Zetafunktionen an ganzzahligen Stellen, *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II* **1969**, 87–102 (1969) [= pages 82–97 in *Gesammelte Abhandlungen IV*, Berlin: Springer 1979].

unique homogeneous polynomial $P(\cdot,\cdot)$ of degree k such that $f^r P(f,g) = \sum_{i=0}^{k} a_j q^j + O(q^{k+1})$; we construct this polynomial iteratively starting from the leading coefficient a_0, as we did when $a_0 = 1$ and $a_1 = \ldots = a_k = 0$. For this choice $(1,0,0,\ldots,0)$ of $a_0,\ldots,a_k$, we find P such that $f^r P(f,g) = 1 + Cq^{k+1} + O(q^{k+2})$. We shall show that C is $(k+r)/(k+1)$ times the $1/q$ coefficient of $f'/(f^r g^{k+1})$. In particular, $C > 0$ for all $k > 0$ if f^{1-r} and $1/g$ have positive coefficients. In our present setting, f^{1-r} is E_4, E_4^2, or E_4^3, and thus has positive coefficients as desired, while the product expansion for Δ gives

$$\frac{1}{\Delta} = q^{-1}\left(\prod_{n=1}^{\infty}(1-q^n)^{-1}\right)^{24} = q^{-1}\left(\prod_{n=1}^{\infty}(1+q^n+q^{2n}+q^{3n}+\cdots)\right)^{24} \tag{3}$$

which manifestly has positive coefficients. Note that if $r = 0$ then for any f,g the $1/q$ coefficient of f'/g^{k+1} is well-defined, even though f could have been changed to $f + Ag$ for any constant A, because this changes f'/g^{k+1} by Ag'/g^{k+1}, which is the derivative of $-Ag^{-k}/(k+1)$ and thus has no $1/q$ term.

To prove our formula, begin by dividing both sides of $f^r P(f,g) = 1 + Cq^{k+1} + O(q^{k+2})$ by $f^r g^k$ to obtain $p(f/g) = 1/(f^r g^k) + Cq + O(q^2)$ where p is the degree-k univariate polynomial $p(X) = P(X,1)$. Since g and g/f are both of the form $q + O(q^2)$, while $1/f_0 = 1 + O(q^2)$, we can find (unique) power series $G(z) = z + b_2 z^2 + b_3 z^3 + \cdots$ and $F(z) = 1 + O(z)$ such that $g = G(g/f)$ and $f = F(g/f)$. (These power series are holomorphic in a neighborhood of $z = 0$ if f and g have positive radii of convergence, but we need F, G only as formal power series.) Taking $z = g/f$, we find $F(z)^{-r}/G(z)^k = p(1/z) - Cz + O(z^2)$, in which the right-hand side is the beginning of the Laurent expansion of $F(z)^{-r}/G(z)^k$ about $z = 0$. Therefore $-C$ is the z coefficient of that expansion. But then C is the residue of $-(F^{-r}/G(z)^k)\,dz/z^2$ at $z = 0$. Now we use the invariance of the residue under locally invertible transformations. [Again this is a theorem about formal power series, though the easiest proof is to use complex analysis, truncating the series if necessary to ensure convergence.] This yields

$$C = -\operatorname{Res}_{q=0} \frac{1}{f^r g^k}\frac{d(g/f)}{(g/f)^2} = +\operatorname{Res}_{q=0} \frac{g\,df - f\,dg}{f^r g^{k+2}}. \tag{4}$$

On the other hand, we have

$$d(f^{1-r}/g^{k+1}) = \frac{(1-r)g\,df - (k+1)f\,dg}{f^r g^{k+2}}, \tag{5}$$

so the residue at $q = 0$ of $\big((1-r)g\,df - (k+1)f\,dg\big)/(f^r g^{k+2})$ vanishes. Multiplying (4) by $k+1$ and subtracting the right-hand side of (5, we find that

$$(k+1)C = (k+r)\operatorname{Res}_{q=0} \frac{df}{f^r g^(k+1)}, \tag{6}$$

whence C is $(k+r)/(k+1)$ times the $1/q$ coefficient of $f'/(f^r g^{k+1})$, as claimed.

It follows that a self-dual even lattice L of rank $n > 0$ has $N_{\min}(L) \le 2\lfloor n/24 \rfloor + 2$. If L attains this bound, it is said to be *extremal*. Our Lemma then shows that if L is an extremal self-dual even lattice then every shell of L is an $11\frac{1}{2}$-design if $n \equiv 0 \bmod 24$, a $7\frac{1}{2}$-design if $n \equiv 8 \bmod 24$, and a $3\frac{1}{2}$-design if $n \equiv 16 \bmod 24$.

We have seen that such L exist for $n = 8$ (E_8), $n = 16$ (E_8^2 and D_{16}^+), and $n = 24$ (Λ_{24}). For $n = 32$, King showed[2] that there are more than 10^7 extremal lattices, so we don't expect to see this list any time soon, let alone the one for $n = 40$. For $n = 48$, we again expect such lattices to be rare; three are known (the two described in SPLAG, and a third found by Nebe[3]), but it is anyone's guess if they are few or plentiful. More recently[4] Nebe constructed an extremal lattice of rank 72. This is the largest $n \equiv 0 \bmod 24$ for which an extremal L of rank n is known, though there are several known examples with $n > 48$ in the congruence classes $8, 16 \bmod 24$.

For very large n (starting around $4 \cdot 10^4$), the extremal theta function has a negative q^{k+2} coefficient, so there can be no extremal theta function. This was proved by Mallows, Odlyzko, and Sloane, who furthermore showed[5] that for every k_0 there is some n_0 such that once $n > n_0$ there is no even self-dual lattice of rank n with $N_{\min} > 2\lfloor n/24 \rfloor - k_0$, because the theta series would have a negative coefficient.

One can likewise define the notion of an extremal self-dual lattice that need not be even, using $f = \theta_{\mathbf{Z}}(q^2)^8$ and $g = \Delta_+(q^2)$ (this Δ_+, like Δ, has a product formula that gives positivity of the coefficients of $1/g$). Here we take $r = \{n/8\}$, and find that $N_{\min}(L) \le \lfloor n/8 \rfloor + 1$. But in this setting we exhaust the "extremal" lattices (that is, self-dual lattices with $N_{\min}(L) = \lfloor n/8 \rfloor + 1$) much sooner: we have already seen $\mathbf{Z}^n$ ($n < 8$), E_8, D_{12}^+, $(E_7^2)^+$, A_{15}^{+4}, and now O_{23} and Λ_{24}; and there are no others.[6]

We conclude with consequences of the spherical-design properties for extremal lattices of ranks 32, 48, and 72.

If L is an extremal lattice of rank 32 then

$$\theta_L = \mathsf{E}_4^4 - 960\,\mathsf{E}_4\Delta = 1 + 146880q^2 + 64757760q^3 + 4844836800q^4 + O(q^5). \tag{7}$$

The 146880 minimal vectors constitute a spherical 7-design. This lets us determine the counts $N_k(v_0)$

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[2] Oliver King: A mass formula for unimodular lattices with no roots, *Math. Comp.* **72**, 839–863 (2003).

[3] See page xlv of the preface to SPLAG 3rd ed.; the reference is Gabriele Nebe: Some cyclo-quaternionic lattice, *J. Alg.* **199**, 472–498 (1998).

[4] Gabriele Nebe: An even unimodular 72-dimensional lattice of minimum 8, *J. reine angew. Math.* **673**, 237–247 (2012). arXiv:1008.2862 (math.NT)

[5] C.L. Mallows, A.M. Odlyzko, and N.J.A. Sloane: Upper Bounds for Modular Forms, Lattices, and Codes. *J. Alg.* **36**, 68–766 (1975).

[6] J. H. Conway, A. M. Odlyzko, and N. J. A. Sloane: Extremal self-dual lattices exist only in dimensions 1 to 8, 12, 14, 15, 23, and 24. Mathematika **25** #1, 36–43 (June 1978). http://neilsloane.com/doc/Me59.pdf

only for $\langle v_0, v_0\rangle = 4$ or 6; already for $\langle v_0, v_0\rangle = 8$ there is one degree of freedom:

$\langle v_0, v_0\rangle$	#	N_0	$N_{\pm1}$	$N_{\pm2}$	$N_{\pm3}$	$N_{\pm4}$
4	$146880 = 2^6 3^3 5\cdot 17$	80910	31744	1240	0	1
6	$64757760 = 2^{13} 3\cdot 5\cdot 17\cdot 31$	66060	35640	4698	72	0
8	$4844836800 = 2^6 3^5 5^2 17\cdot 733$	$70N_4 + 57040$	$36240 - 56N_4$	$28N_4 + 8184$	$496 - 8N_4$	N_4

Unfortunately neither the "$7\frac{1}{2}$-design" refinement, nor the positivity of the N_k, suffices to determine $N_4(v_0)$ for $\langle v_0, v_0\rangle = 8$. We already know for geometrical reasons that $N_4(v_0)$ must be some nonnegative even number and that $N_4(v_0) \le 2(32-1) = 62$; and all 32 possible choices yield nonnegative N_k, though $N_4 = 62$ makes $N_3 = 0$. This does happen, if we apply Leech's construction to a suitable self-dual doubly even code of length 32; but this is quite unusual: few L are of this form, and for any extremal L of rank 32, the average of $N_4(v_0)$ over all v_0 of norm 8 is only

$$146880 \cdot 80910/4844836800 = 1798/733 < 2.453.$$

If L is an extremal lattice of rank 48 then

$$\begin{aligned} \theta_L &= \mathsf{E}_4^6 - 1440\,\mathsf{E}_4^3\Delta + 125280\Delta^2 \\ &= 1 + 52416000q^3 + 39007332000q^4 + 6609020221440q^5 + O(q^6); \end{aligned} \tag{8}$$

The counts of vectors of norm $6, 8, 10$ factor as

$$\begin{aligned} 52416000 &= 2^9\, 3^2\, 5^3\, 7 \cdot 13, \\ 39007332000 &= 2^5\, 3^7\, 5^3\, 7^3\, 13, \\ 6609020221440 &= 2^{11}\, 3^8\, 5 \cdot 7 \cdot 13 \cdot 23 \cdot 47, \end{aligned} \tag{9}$$

and the counts $N_k(v_0)$ for these three cases are

$\langle v_0, v_0\rangle$	N_0	$N_{\pm1}$	$N_{\pm2}$	$N_{\pm3}$	$N_{\pm4}$	$N_{\pm5}$	$N_{\pm6}$
6	23766960	12608784	1678887	36848	0	0	1
8	20582240	12816384	2905728	192512	2256	0	0
10	18409300	12612600	3898200	475300	17150	100	0

For $\langle v_0, v_0\rangle = 12$, the average $N_6 = N_6(v_0)$ is

$$\frac{52416000 \cdot 23766960}{437824977408000} = \frac{990290}{348037} = 2.845+,$$

and $N_0, N_{\pm1}, N_{\pm2}, N_{\pm3}, N_{\pm4}, N_{\pm5}$ are

$$16802688 + 924N_6,\ 12270384 - 792N_6,\ 4642848 + 495N_6,\ 833592 - 220N_6,\ 58656 + 66N_6,\ 1176 - 12N_6,$$

with $N_{\pm5} = 12(98 - N_6)$. Using also the $11\frac{1}{2}$-design condition, we likewise analyze $\langle v_0, v_0\rangle = 14$: here $N_7(v_0)$ is either 0 or 2, with the latter occurring for

$$\frac{1}{2}52416000 \cdot 12608784 = 330451011072000$$

choices of v_0, which is $12/551 < 2.2\%$ of the number $N_{14}(L) = 15173208925056000$ of lattice vectors of norm 14; we then find that the possible counts $(N_0, N_{\pm 1}, N_{\pm 2}, \cdots, N_{\pm 7})$ are

$$\begin{array}{l}(15558368, 11883840, 5190387, 1215048, 133992, 5496, 53, 0),\\ (15573680, 11870838, 5198263, 1211770, 134848, 5390, 49, 2).\end{array} \tag{10}$$

For extremal L of rank 72,

$$\begin{aligned}\theta_L &= \mathsf{E}_4^9 - 2160\,\mathsf{E}_4^6\Delta + 965520\,\mathsf{E}_4^3\Delta^2 - 27302400\Delta^3 \\ &= = 1 + 6218175600q^4 + 15281788354560q^5 + 9026867482214400q^6 + O(q^7),\end{aligned} \tag{11}$$

with factorizations

$$6218175600 = 2^4\, 3^5\, 5^2\, 7 \cdot 13 \cdot 19 \cdot 37, \quad 15281788354560 = 2^{16}\, 3^6\, 5 \cdot 7 \cdot 13 \cdot 19 \cdot 37, \ldots;$$

the nonzero counts $N_k(v_0)$ are

$$(N_0, N_{\pm 1}, N_{\pm 2}, N_{\pm 3}, N_{\pm 4}, N_{\pm 8}) = (2603658750, 1512243200, 280928256, 13959168, 127800, 1)$$

for $\langle v_0, v_0\rangle = 8$, and

$$(N_0, N_{\pm 1}, N_{\pm 2}, N_{\pm 3}, N_{\pm 4}, N_{\pm 5}) = (2328777990, 1508892000, 396819000, 37926000, 1056125, 5680)$$

for $\langle v_0, v_0\rangle = 10$.

For the numbers $565866362880, 45792819072000, \ldots$ of minimal vectors in putative extremal lattices of dimensions $n = 96, 120, \ldots$, see OEIS Sequence 34597 (http://oeis.org/A034597). By the time we reach $n = 96$ the 11-design property gives the counts $N_k(v_0)$ only for minimal vectors v_0: they are

$$219453729516, 137524268000, 32881785375, 2733804000, 66118250, 341056$$

for $k = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$ respectively, and of course 1 for $k = \pm 10$ and zero otherwise.