Math 272y: Rational Lattices and their Theta Functions

2 December 2019: Shifted theta functions $\theta_{L+v_0,P}$ with $v_0 \in L^*$,
and their modularity for $\operatorname{disc} L \le 5$

We do not have enough time left in the semester to prove in full generality the modularity of

$$\sum_{v \in L} f(v) P(v) \exp(\pi i \langle v, v \rangle z) \tag{1}$$

for every rational lattice $L \in \mathbf{R}^n$, harmonic polynomial P, and periodic function $f : L \to \mathbf{C}$. We have already reduced this general result to the special case that $L = L_0^*$ for some even lattice L_0 that is contained in the period lattice of f. Today we give the next few steps, describe how we would complete the proof, and work out a proof and an application in the special (but still new and simpler) case that $\operatorname{disc} L_0 = 5$.

From here on we work mainly in terms of the period lattice, so we switch the roles of L and L_0. Thus L will be an even lattice, and (1) is a linear combination of weighted theta functions of the lattice translates $L + v_0$ with $v_0 \in L^*$, which we naturally call

$$\theta_{L+v_0,P} := \sum_{v \in L+v_0} P(v) \exp(\pi i \langle v, v \rangle z) = \sum_{v \in L+v_0} P(v) q^{\langle v,v \rangle/2} \tag{2}$$

where $q = e^{2\pi i z}$ as usual and fractional powers q^n ($n \in \mathbf{Q}$) are interpreted as $e^{2\pi i n z}$. We already outlined the general strategy: fix L and P, let v_0 vary over representatives of the discriminant group L^*/L, and work out the action of the $\mathrm{PSL}_2(\mathbf{Z})$ generators $S : z \mapsto -1/z$ and $T : z \mapsto z + 1$ on all functions $\theta_{L+v_0,P}$ simultaneously. We soon see that T multiplies each $\theta_{L+v_0,P}$ by the scalar $\exp(\pi i \langle v_0, v_0 \rangle)$, while S takes each $\theta_{L+v_0,P}$ to z^k times some linear combination of all the $\theta_{L+v_0',P}$, where k is the usual weight $(n/2) + \deg P$. Since S and T generate $\Gamma(1)$, it follows that for every $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$ each $\theta_{L+v_0,P}$ is $(cz+d)^k$ times some linear combination of all the $\theta_{L+v_0',P}$. We combine all the $\theta_{L+v_0,P}$ into a single vector-valued function $\boldsymbol{\theta}_{L,P} : \mathcal{H} \to \mathbf{C}^D$ (where $D = \operatorname{disc} L$), and show that

$$\boldsymbol{\theta}_{L,P}(gz) = (cz+d)^k \rho(g) \boldsymbol{\theta}_{L,P}(z) \tag{3}$$

for some $\rho(g) \in \mathrm{GL}_D(\mathbf{C})$. Here $\rho : \Gamma(1) \to \mathrm{GL}_D(\mathbf{C})$ is a representation that depends only on n, $\deg P$, and the discriminant form on L^*/L. Note that it is not quite automatic that ρ is a homomorphism: the matrices $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ that represent S and T generate $\mathrm{SL}_2(\mathbf{Z})$, but not freely, so we must check that ρ respects the relations coming from $S^2 = (ST)^3 = 1$. Once we do this, we shall see that moreover changing n and $\deg P$ changes each $\rho(g)$ only by some scalar factor, so the projective representation $\rho/\mathbf{C}^* : \Gamma(1) \to \mathrm{PGL}_D(\mathbf{C})$ depends only on the discriminant form.

Now the key fact is that for $2|n$ this representation "factors through" some congruence subgroup: there exists N (namely the level of L) such that ρ is trivial on $\Gamma(N)$, and thus descends to a representation of the finite group $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$. This makes $\boldsymbol{\theta}_{L,P}$ a "vector-valued modular form" of weight k for this representation; it follows that each $\theta_{L+v_0,P}$ is a modular form of weight k for $\Gamma(N)$. Note that we write "modular" and not just "weakly modular": each $\theta_{L+v_0,P}$ is automatically bounded at the cusp $z = i\infty$, and since the full modular group $\Gamma(1)$ acts on $\boldsymbol{\theta}_{L,P}$ this takes care of all the other cusps too.

If n is odd, then k is half-integral, so ρ cannot factor through any congruence subgroup, though $\rho/\mathbf{C}^*$ still does. But then $\theta_{\mathbf{Z}}\,\boldsymbol{\theta}_{L,P}$ satisfies a transformation formula (3) for all $g \in \Gamma_+$, with the exponent k replaced by $k+\frac{1}{2} \in \mathbf{Z}$ and each $\rho(g)$ multiplied by an 8th root of unity $\epsilon_{c,d}$. This new transformation formula does factor through some congruence group of Γ_+, making $\theta_{\mathbf{Z}}\,\boldsymbol{\theta}_{L,P}$ a vector-valued modular form of weight $k+\frac{1}{2}$. We then say that $\boldsymbol{\theta}_{L,P}$ itself is a vector-valued modular form of half-integral weight k.

We next determine the images under ρ of the matrices $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ that represent S and T. For the T matrix is easy: given $v_0 \in L^*$, we know that all $v \in L+v_0$ have the same $\langle v,v\rangle \bmod 2$; thus changing z to $z+1$ multiplies each $\theta_{L+v_0,P}$ by the scalar $\exp(\pi i \langle v_0, v_0\rangle)$. We thus take $\rho(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))$ to be the diagonal matrix representing this transformation. Note that $\rho(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))^N$ is the identity, as it should be because $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)^N = \left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma(N)$. For the S matrix, we need $\theta_{L+v_0,P}(-1/z)$. Recall that we proved the Poisson summation formula from the Fourier series of $\sum_{v\in L+v_0} f(v)$ as an L-periodic function of $v_0 \in \mathbf{R}^n$:, for any Schwartz function $f : \mathbf{R}^n \to \mathbf{C}$:

$$\sum_{v\in L+v_0} f(v) = (\operatorname{disc} L)^{-1/2} \sum_{w\in L^*} \hat{f}(-w) e^{2\pi i \langle v_0, w\rangle}. \tag{4}$$

We apply this formula to $f(v) = P(v)\exp(\pi i \frac{-1}{z}\langle v,v\rangle)$. For once we must be careful to have consistent signs and normalizations in our definitions and formulas involving the Fourier transform. Taking $d = \deg P$, we proved that the Fourier transform of $P(x)\exp(-\pi\langle x,x\rangle t)$ is

$$i^d t^{-(\frac{n}{2}+d)} P(y)\, e^{-\pi\langle y,y\rangle/t}, \tag{5}$$

first for $t > 0$ and then by analytic continuation for all t of positive real part (that is, with $it \in \mathcal{H}$). Taking $t = i/z$, we deduce that our f has Fourier transform

$$i^d (z/i)^{\frac{n}{2}+d} P(y)\, e^{\pi i z\langle y,y\rangle t}. \tag{6}$$

Substituting this into (4) yields

$$\theta_{L+v_0,P}(-1/z) = (\operatorname{disc} L)^{-1/2} (z/i)^{n/2} z^d \sum_{w\in L^*} e^{-2\pi i\langle v_0,w\rangle} P(w) e^{\pi i\langle w,w\rangle z}. \tag{7}$$

In our setting $v_0 \in L^*$, so $e^{2\pi i\langle v_0,w\rangle}$ is a root of unity depending only on $w \bmod L$. Hence

$$\theta_{L+v_0,P}(-1/z) = (\operatorname{disc} L)^{-1/2} (z/i)^{n/2} z^d \sum_{v'\in L^*/L} e^{-2\pi i(v_0,v')} \sum_{w\in L+v'} P(w) e^{\pi i\langle w,w\rangle z}, \tag{8}$$

which is to say

$$\theta_{L+v_0,P}(-1/z) = (\operatorname{disc} L)^{-1/2} (z/i)^{n/2} z^d \sum_{v'\in L^*/L} e^{-2\pi i(v_0,v')} \theta_{L+v',P}(z), \tag{9}$$

where $(\cdot,\cdot)$ is the symmetric $(\mathbf{Q}/\mathbf{Z})$-valued pairing on L^*/L. Thus we take for $\rho(S)$ a multiple of the discrete Fourier transform on L^*/L. Our usual Poisson check of applying (9) a second time yields

$$\theta_{L+v_0,P}(z) = (\operatorname{disc} L)^{-1}(-1)^d \sum_{v''}\Big[\sum_{v'} e^{-2\pi i(v_0+v'',v')}\Big]\theta_{L+v'',P}(z), \tag{10}$$

with the sums over v'' and v' both ranging over L^*/L; the bracketed sum gives $\#(L^*/L) = \operatorname{disc} L$ if $v_0 + v'' = 0$, and zero otherwise, so (10) recovers the identity

$$\theta_{L-v_0,P} = (-1)^d \theta_{L+v_0,P}. \tag{11}$$

Note that ρ thus maps the central involution $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)^2 = -I_2 \in \mathrm{SL}_2(\mathbf{Z})$ *not* to the identity (unless $N \le 2$), but to a permutation matrix of order 2 or its negative; but either way this image commutes with both $\rho(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))$ and $\rho(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))$, and thus with all of $\rho(\mathrm{SL}_2(\mathbf{Z}))$.

To complete the modularity proof we would have to check that $\rho(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right))^3$ maps to the same involution, so that ρ is a homomorphism, and than that ρ maps $\Gamma(N)$ to the identity. The former is an exercise but a somewhat lengthy one[1] that we omit for lack of time. The latter requires one or two more ideas. We have seen already that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)^N = \left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right)$ maps to the identity; this would be enough if the $\mathrm{SL}_2(\mathbf{Z})$ conjugates of $\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right)$ generated $\Gamma_1(N)$, or equivalently if the group

$$\langle \sigma, \tau | \sigma^2 = (\sigma\tau)^3 = \tau^N = 1 \rangle \tag{12}$$

(which is the quotient of $\mathrm{PSL}_2(\mathbf{Z})$ by the subgroup generated by those conjugates) were the finite group $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})/\{\pm 1\}$. But (12) is a triangle group, and is thus finite if and only if it is spherical, which happens only for $N < 6$ (which is equivalent to the criterion $1/2 + 1/3 + 1/N > 1$). If $N \ge 6$, we can still verify for any given discriminant form that ρ is a homomorphism by computing its image on each element of the finite group $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})/\{\pm 1\}$ and checking that $\rho(g)\rho(g') = \rho(gg')$ for all g, g' in that group; but this approach cannot prove the result in general, though the explicit matrices $\rho(g)$ might suggest the general structure of ρ. For the general proof, one constructs a representation $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z}) \to \mathrm{GL}_D(\mathbf{C})$ some other way, and checks that the images of $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ agree with those of ρ. One approach is via the Weil representation, which normalizes the finite Heisenberg group of linear transformations of $\mathbf{C}^D$ generated by translations and characters of L^*/L.

We content ourselves with noting that for $N < 6$ the spherical triangle group (12) does coincide with $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})/\{pm1\}$. This is clear for $N = 1$, when both groups are trivial, and easy for $N = 2$ when each group is isomorphic with the symmetric group S_3. For $N = 3, 4, 5$ these are the tetrahedral, octahedral, and icosahedral groups A_4, S_4, A_5. In each case, once we check that $\rho(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right))^3 = \rho(I_2)$, we obtain the modularity of all the shifted theta functions $\theta_{L+v_0,P}$ of a level-N lattice L. This is new even for $2 \le N \le 4$, for which we so far knew only the special case $v_0 = 0$ (and the modularity of $\theta_{L^*,P} = \sum_{v_0 \in L^*/L} \theta_{L+v_0}(P)$). For example, for $L = \Lambda_{16}$ and $v_0 \ne 0$ we find that θ_{L+v_0} depends only on whether $Q(v_0) = 0$ or $Q(v_0) = 1$; and for $L = E_6$ and $v_0 \ne 0$ we can deduce that each shell of $L + v_0$, notably including its 27 minimal vectors, constitute a spherical 4-design. We conclude this chapter of the notes by working out an example with $N = 5$, for which we did not previously know even that θ_L is modular.

We assume that $\operatorname{disc} L = 5$, so that $\rho(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right))^3 = \rho(I_2)$ is an identity between square matrices of order 5 (rather than 25 or more). There are two possibilities for the discriminant form; we find that (as we have seen for other levels) the choice depends on $n \bmod 8$:

Proposition. *Let $L \subset \mathbf{R}^n$ be a lattice of discriminant 5. Then $4|n$, and the quadratic form $[v] \mapsto (5\langle v, v \rangle) \bmod 5$ on L^* takes square or non-square values according as $n \equiv 4$ or $0 \bmod 8$.*

[1] It also requires at one point the sign of the Gauss sum, as we shall see when we work out the case $\operatorname{disc} L = 5$.

Proof: That $4|n$ is a special case of our result that an even lattice of discriminant $1 \bmod 4$ has rank $0 \bmod 4$. The second part will follow by a gluing argument once we find a single example of each case, because -1 is a square $\bmod 5$. For $n \equiv 4 \bmod 8$, use A_4, which has dual vectors of norms $4/5$ and $6/5$. For $n \equiv 0 \bmod 8$, use the lattice $(E_7 \oplus \mathbf{Z}\langle 10 \rangle)^+$, which has dual vectors of norms $2 \cdot 8/10 = 8/5$ and $4 \cdot 6/10 = 12/5$. $\square$

Since n is always $0 \bmod 4$, the factor $(z/i)^{n/2} z^d$ in (9) can be written as $(-1)^{(n/4)+d} z^{n/2}$. We next determine the images under ρ of our generators of $\mathrm{SL}_2(\mathbf{Z})$. Let $\zeta = e^{2\pi i/5}$ be the standard generator of the fifth roots of unity. If $n \equiv 4 \bmod 8$, let v_1 be a generator of L^*/L such that $q(v_1) = 4/5$; for example, if $L = A_4$, we can take v_1 to be the coset of $(1,1,1,1,-4)/5$. Identify the functions on L^*/L with $\mathbf{C}^5$ using the order $0, v_1, 2v_1, 3v_1, 4v_1$ on L^*/L. Then $\rho(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)) = \mathrm{diag}(1, \zeta^2, \zeta^3, \zeta^3, \zeta^2)$ and $\rho(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))$ is

$$(-1)^{d+1} 5^{-1/2} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 & \zeta^3 & \zeta^4 \\ 1 & \zeta^2 & \zeta^4 & \zeta & \zeta^3 \\ 1 & \zeta^3 & \zeta & \zeta^4 & \zeta^2 \\ 1 & \zeta^4 & \zeta^3 & \zeta^2 & \zeta \end{pmatrix}. \tag{13}$$

We then compute that $\rho(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right))^3$ is

$$(-1)^{d+1} \frac{(1 + 2\zeta^2 + 2\zeta^3)}{\sqrt{5}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \tag{14}$$

and the fraction is -1, so we get either the involution $\rho(-I_2)$ or its negative according as d is even or odd, consistent with (11) in either case. If $n \equiv 0 \bmod 8$, we take for v_1 a generator of L^*/L such that $q(v_1) = 2/5$. Then $\rho(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)) = \mathrm{diag}(1, \zeta, \zeta^4, \zeta^4, \zeta)$ and $\rho(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))$ is

$$(-1)^{d} 5^{-1/2} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \zeta^3 & \zeta & \zeta^4 & \zeta^2 \\ 1 & \zeta & \zeta^2 & \zeta^3 & \zeta^4 \\ 1 & \zeta^4 & \zeta^3 & \zeta^2 & \zeta \\ 1 & \zeta^2 & \zeta^4 & \zeta & \zeta^3 \end{pmatrix}, \tag{15}$$

both obtained from the previous formulas by Galois automorphism that takes ζ^2 to ζ, and thus $\sqrt{5}$ to $-\sqrt{5}$. This gives the same final result for $\rho(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right))^3$, this time with the coefficient appearing as $(-1)^d(1 + \zeta + \zeta^4)/5$.

Since $\rho(-I_2)$ commutes with the image of ρ, the $+1$ and -1 eigenspaces of its action on $\mathbf{C}^5$ are subrepresentations of ρ. Both of these subspaces show the identification of $\mathrm{PSL}_2(\mathbf{Z}/5\mathbf{Z})$ with the isometries of the icosahedron: the $+1$ subspace is the 3-dimensional icosahedral representation of $\mathrm{SL}_2(\mathbf{Z}/5\mathbf{Z})/\{\pm 1\} \cong A_5$, defined over the real subfield $\mathbf{Q}(\sqrt{5})$ of $\mathbf{Q}(\zeta)$; and the -1 subspace is the 3-dimensional representation of $\mathrm{SL}_2(\mathbf{Z}/5\mathbf{Z})$, whose projectivization yields the action of A_5 on the Riemann sphere, again as the orientation-preserving symmetries of a regular icosahedron. These spaces appear in the action of $\mathrm{SL}_2(\mathbf{Z}/5\mathbf{Z})$ on theta functions weighted by harmonic polynomials of even and odd degree respectively.

When $n \equiv 4 \bmod 8$, we can recover the Nebentypus of θ_L by computing the action of an $\mathrm{SL}_2(\mathbf{Z})$ element $g \in \Gamma_1(5)$ such as $\left(\begin{smallmatrix} 2 & 1 \\ 5 & 3 \end{smallmatrix}\right)$ whose diagonal entries generate $(\mathbf{Z}/5\mathbf{Z})^*$. The corresponding element of $\mathrm{PSL}_2(\mathbf{Z})$ is $ST^{-1}STST^{-2}S$, and its image is the permutation matrix

$$-\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta & 0 \\ 0 & \zeta^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta^4 \\ 0 & 0 & \zeta & 0 & 0 \end{pmatrix}. \tag{16}$$

In particular, the first column gives $\theta_L(gz) = -(5z+3)^{n/2}\theta_L(z)$. Therefore the Nebentypus character is χ_5.

Now suppose $L = A_4$. There are no nonzero cusp forms of weight 2 for $\Gamma_0(5)$ (the first cuspform is $(\eta(z)\eta(5z))^4$, of weight 4). Therefore we can write θ_{A_4} as a linear combination of the Eisenstein series[2] $G_2(z,\chi_5)$ and $w_5 G_2(z,\chi_5)$, with coefficients determined by $\operatorname{disc} L$. Here we find that

$$\theta_{A_4}(z) = 1 + 20q + 30q^2 + 60q^3 + 60q^4 + 120q^5 + 40q^6 + 180q^7 + 150q^8 + \cdots, \tag{17}$$

$$\theta_{A_4^*}(z) = \theta_{A_4}(z) + 10q^{2/5} + 20q^{3/5} + 60q^{7/5} + 50q^{8/5} + 60q^{12/5} + 120q^{13/5} + \cdots \tag{18}$$

have coefficients

$$N_{2n}(A_4) = \sum_{d|n} \big(25\chi_5(n/d) - 5\chi_5(d)\big) d \tag{19}$$

and

$$N_{2n/5}(A_4^*) = \sum_{d|n} \big(5\chi_5(n/d) - 5\chi_5(d)\big) d. \tag{20}$$

(Check that (20) vanishes for $\chi_5(n) = +1$, as it must.) The theta functions $\theta_{A_4 \pm v_1}$ and $\theta_{A_4 \pm 2v_1}$ can be recovered from $\theta_{A_4^*}$ as $\frac{1}{2}$ times the sums of the terms whose exponents are congruent mod 2 to $4/5$ and $6/5$ respectively:

$$\theta_{A_4 \pm v_1} = 5q^{2/5} + 30q^{7/5} + 30q^{12/5} + \cdots, \quad \theta_{A_4 \pm 2v_1} = 10q^{3/5} + 25q^{8/5} + 60q^{13/5} + \cdots. \tag{21}$$

For a further check, we reconstruct the theta function of $\mathbf{Z}^5$ from the shifted theta functions of A_4 and its orthogonal complement $\mathbf{Z}\sum_{j=1}^5 e_j \cong \mathbf{Z}\langle 5 \rangle$, and compare with $\theta_{\mathbf{Z}}^5$. Each $\mathbf{Z}^5$ vector w is uniquely $v + (c/5)\sum_{j=1}^5 e_j$ for some $v \in A_4^*$ and $c \in \mathbf{Z}$ with $v \in A_4 + cv_1$. Since the sum is orthogonal, w has norm $\langle v, v \rangle + c^2/5$, whence

$$\theta_{\mathbf{Z}}^5 = \theta_{\mathbf{Z}^5} = \sum_{c \bmod 5} \theta_{(\mathbf{Z}+(c/5))\langle 5 \rangle}\, \theta_{A_4 + cv_0}, \tag{22}$$

and indeed both sides give the same q-expansion $1 + 10q^{1/2} + 40q + 80q^{3/2} + 90q^2 + 112q^{5/2} + \cdots$.

[2] The formulas we gave for $G_k(z,\chi)$ converge only for $k > 2$. When the character χ is nontrivial, it is not too hard to extend the construction also to $k = 2$, with the same formulas for the q-expansions. Such series also exist for $k = 1$, though this requires more work.