Coxeter-Dynkin diagrams for ADE and affine ADE lattices

Type A. The root lattice A_n is the slice $\sum_{i=1}^{n+1} a_i = 0$ of the lattice

$$\mathbf{Z}^{n+1} = \Big\{ \sum_{i=1}^{n+1} a_i e_i \,\Big|\, \text{each } a_i \in \mathbf{Z} \Big\}.$$

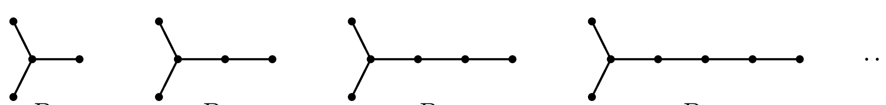
It has rank n and discriminant $n+1$, with $A_n^*/A_n \cong \mathbf{Z}/(n+1)\mathbf{Z}$. The roots are $e_i - e_j$ for distinct $i, j \in [1, n+1]$. The Coxeter-Dynkin diagram is simply a path of length n:

A_1 A_2 A_3 A_4 A_5 etc.

We may take the i-th vertex of the path ($1 \leq i \leq n$) to correspond to the root $e_i - e_{i+1}$.

An extended A_n diagram is formed by attaching the $(n+1)$st vertex to the endpoints of the path. Thus the extended diagram is a cycle of length $n+1$. Each component has multiplicity 1. For $n = 1$ the cycle degenerates to two vertices linked by a double edge. For the extended diagram $\widetilde{A}_n$, we also allow $n = 0$, when the cycle degenerates to a loop. For both $n = 0$ and $n > 0$, the $(n+1)$st vertex corresponds to the root $e_{n+1} - e_1$.

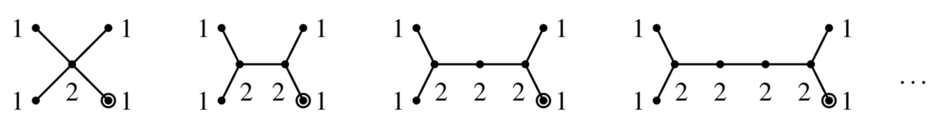
Type D. We have already introduced the lattice D_n, of rank n and discriminant 4. This is a root lattice for $n \geq 2$, with roots $\pm e_i \pm e_j$ for distinct $i, j \in [1, n]$. The D_n for $n \geq 4$ constitute a family of irreducible root lattices distinct from the A_n family. (For $n = 1$ there are no distinct i, j, so D_1 has no roots. That lattice is isomorphic with $2\mathbf{Z} \cong \mathbf{Z}\langle 4 \rangle \cong A_1\langle 2 \rangle$, as noted already; the lattices D_2 and D_3 are isomorphic with A_1^2 and A_3 respectively.) The Coxeter-Dynkin diagram is a path of length $n-2$ with two vertices connected to one of its endpoints:



D_4 D_5 D_6 D_7 etc.

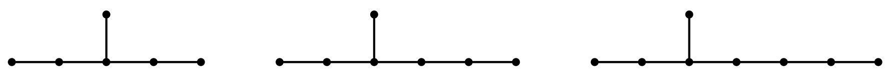
(Note that $D_3 \cong A_3$ and $D_2 \cong A_1^2$ also fit into this pattern.) We may take the i-th node of the path ($1 \leq i \leq n-2$) to correspond to the root $e_{i+2} - e_{i+1}$, with the other two roots joined to $e_3 - e_2$ being $e_2 \pm e_1$. The four-element group D_n^*/D_n is isomorphic with the Klein group $(\mathbf{Z}/2\mathbf{Z})^2$ for n even, and with the cyclic group $\mathbf{Z}/4\mathbf{Z}$ for n odd.

For $n > 4$ the extended D_n diagram consists of a path of length $n-3$, consisting of components each of multiplicity 2, with two vertices attached to each endpoint that correspond to the components of multiplicity 1. For $n = 4$ the path collapses to a single vertex, regarded as both the initial and final endpoints. For both $n = 4$ and $n > 4$, the $(n+1)$st vertex corresponds to the root $-e_{n-1} - e_n$. The next picture shows the first few $\widetilde{D}_n$ diagrams and their component multiplicities:



$\widetilde{D}_4$ $\widetilde{D}_5$ $\widetilde{D}_6$ $\widetilde{D}_7$ etc.

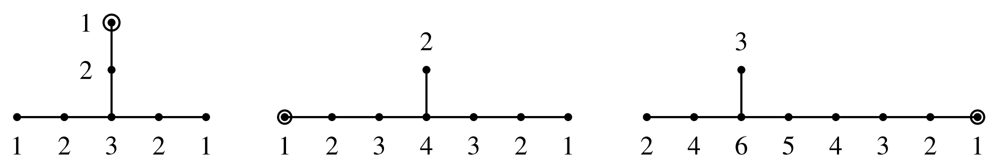
Type E. We have already seen the root lattice $E_8 = D_8^+$. The remaining root lattices E_7 and E_6 can be obtained from E_8 as the slices in which the last two or last three coordinates are equal. The Coxeter-Dynkin diagrams are:



E_6 E_7 E_8

It can be seen that for $r = 6, 7, 8$ the E_r diagram contains a copy of the D_{r-1} diagram. For E_8 we may identify the roots in the D_7 subdiagram with $e_2 \pm e_1$ and $e_{i+2} - e_{i+1}$ as before, and then the extra root is $-\frac{1}{2}\sum_{i=1}^{8} e_i$. We may then delete the root $e_7 - e_6$ to obtain the E_7 diagram, and further delete the root $e_6 - e_5$ to obtain the E_6 diagram, in either case recovering our description of E_7 or E_6 as a slice of E_8. The lattice E_r has discriminant $9 - r$, so $E_8^* = E_8$ while E_7^*/E_7 and E_6^*/E_6 are groups of order 2 and 3.

The extended E_6, E_7 and E_8 diagrams and their component multiplicities are:



$\widetilde{E}_6$ $\widetilde{E}_7$ $\widetilde{E}_8$

Using the same coordinates in which we described the simple roots, the root corresponding to the $(r+1)$st vertex of E_r is $e_8 - e_7$, $e_8 + e_7$, and

$$(e_1 - e_2 - e_3 - e_4 - e_5 + e_6 + e_7 + e_8)/2$$

for $r = 8$, $r = 7$, and $r = 6$ respectively.